THE UNSOLVABILITY OF MULTIPLICATIVE INVERSE EIGENVALUE PROBLEMS ALMOST EVERYWHERE*

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Abstract

The idea and technique used in [7] are applied to the multiplicative inverse eigenvalue problems as well. Some sufficient and necessary conditions that the multiplicative inverse eigenvalue problems be unsolvable almost everywhere are given. The results are similar to those of [7], but the proofs are more complicated.

§ 1. Introduction

The multiplicative inverse eigenvalue problems for real matrices are the following (see [2], [4]):

Problem M-1. Given an $n\times n$ positive definite symmetric matrix A, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real $n\times n$ diagonal matrix $C=\mathrm{diag}(c_1, \cdots, c_n)$ such that the matrix CA has a zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

Problem GM-1. Given m real $n\times n$ symmetric matrices $A_1, \cdots, A_m$, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find m real numbers $c_1, \cdots, c_m$ such that the matrix $c_1A_1+\cdots+c_mA_m$ has a zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

Problem M-2. Given a real $n\times n$ nonsingular matrix A, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real $n\times n$ diagonal matrix $C=\mathrm{diag}(c_1, \cdots, c_n)$ such that the matrix CA is diagonalizable and has a zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

Problem GM-2. Given m real $n\times n$ matrices $A_1, \cdots, A_m$, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find m real numbers $c_1, \cdots, c_m$ such that the matrix $c_1A_1+\cdots+c_mA_m$ is diagonalizable and has a zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

Problem M-1 is a classical multiplicative inverse eigenvalue problem (see [4]). Problems GM-1 and GM-2 are general multiplicative inverse eigenvalue problems for real matrices (see [2]). These problems have been studied by several authors,

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This paper is a continuation of [7]. In this paper we give some sufficient and necessary conditions that the problems M-1, GM-1, M-2 and GM-2 be unsolvable almost everywhere (a.e.), respectively.

Notation. The symbol $\mathbb{R}^{m\times n}$ denotes the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. $I^{(n)}$ is the $n\times n$ identity matrix and O is the null matrix. $\mathscr{R}(A)$ stands for the column space of A. The superscript T is for transpose, and

$$S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^T=A\},\ \mathbb{O}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^TA=I\}$$

and

$$S\mathbb{R}_+^{n\times n}=\{A\in S\mathbb{R}^{n\times n}:\ A \text{ is positive definite}\}.$$

Besides, for $A=(a_{ij})\in\mathbb{R}^{n\times n}$ we write

$$|A|=(|a_{ij}|),\ k_1(A)=\max_{1<j<n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|\Big)$$

and

$$k_2(A)=\max_{1<j<n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}a_{ij}^2\Big)^{1/2}.$$

Now we define the unsolvability of multiplicative inverse eigenvalue problems a.e.

Definition 1.1. *Problem M-1 is said to be unsolvable almost everywhere (u.s.a.e.) if the set of matrices* $A\in S\mathbb{R}_+^{n\times n}$ *and vectors* $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ *at which it is solvable has measure zero in the open set* $S\mathbb{R}_+^{n\times n}\times\mathbb{R}^k$ *of the product vector space* $S\mathbb{R}^{n\times n}\times\mathbb{R}^k$.

Definition 1.2. *Problem GM-1 is said to be u.s.a.e. if the set of matrices* $A_1,\cdots,A_m\in S\mathbb{R}^{n\times n}$ *and vectors* $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ *at which it is solvable has measure zero in the product vectors space* $\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k$.

Definition 1.3. *Problem M-2 is said to be u.s.a.e. if the set of matrices* $A\in\mathbb{R}^{n\times n}$ *and vectors* $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ *at which it is solvable has measure zero in the product vector space* $\mathbb{R}^{n\times n}\times\mathbb{R}^k$.

Definition 1.4. *Problem GM-2 is said to be u.s.a.e. if the set of matrices* $A_1,\cdots,A_m\in\mathbb{R}^{n\times n}$ *and vectors* $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ *at which it is solvable has measure zero in the product vector space* $\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k$.

§ 2. Main Results

Theorem 2.1. *Problem M-1 is u.s.a.e. if and only if*

$$\max\{r_1,\cdots,r_k\}\geq 1. \tag{2.1}$$

Theorem 2.2. *Problem GM-1 is u.s.a.e. if*

$$n-m+\frac{r(r-1)}{2}\geq 0, \tag{2.2}$$

where $r=\max\{r_0,r_1,\cdots,r_k\}$. *In addition, if* $m=n$, *then* $r>1$ *is a sufficient and necessary condition for the unsolvability of Problem GM-1 a.e.*

Theorem 2.3. *Problem M-2 is u.s.a.e. if and only if*