A THIRD ORDER SMALL PARAMETER METHOD AND ITS NORDSIECK EXPRESSION*

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Abstract

A third order small parameter method and its Nordsieck expression are given in this paper. It is based on Gear's method of order 2 and order 3. For moderate stiff problems this method is suitable.

In [1] we proposed a second order numerical method for stiff ODEs. The purpose of this paper is to raise the order from 2 to 3 and give its Nordsieck expression, making it automatically suit varying stepsize calculation.

§ 1. Derivation of the Method and the Truncation Error

For the differential equation

$$y' = f(t, y), \tag{1.1}$$

from the second order Gear formula

$$y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2}{3} h f_{n+1}, \tag{1.2}$$

we have

$$\frac{2}{3} h y'_{n+1} = y_{n+1} - \frac{4}{3} y_n + \frac{1}{3} y_{n-1} \tag{1.3}$$

and from (1.1) we have

$$\varepsilon y'_{n+1} = \varepsilon f_{n+1}, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter.

(1.3) + (1.4) yields

$$\frac{2}{3} h y'_{n+1} = p\left[\varepsilon f_{n+1} + y_{n+1} - \frac{4}{3} y_n + \frac{1}{3} y_{n-1} \right], \tag{1.5}$$

where

$$p = \frac{h}{h + \frac{3}{2}\varepsilon}, \quad 0 < p < 1.$$

We rewrite the third order Gear formula as follows:

$$y_{n+1} = \frac{18}{11} y_n - \frac{9}{11} y_{n-1} + \frac{2}{11} y_{n-2} + \frac{6}{11} \cdot \frac{3}{2} \left(\frac{2}{3} h y'_{n+1} \right). \tag{1.6}$$

From (1.5) and (1.6), we have

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$$y_{n+1}=\frac{18}{11}y_n-\frac{9}{11}y_{n-1}+\frac{2}{11}y_{n-2}+\frac{9}{11}p\Big[\varepsilon f_{n+1}+y_{n+1}-\frac{4}{3}y_n+\frac{1}{3}y_{n-1}\Big]. \quad (1.7)$$

Expanding both sides of (1.7) in the Taylor expression form, we get

$$y_n'=f_n-\frac{4}{3}\frac{1}{\varepsilon}\cdot\frac{h^3}{6}y_n'''+O(h^4). \quad (1.8)$$

Then the local truncation error is

$$\frac{2}{9}\frac{h^4}{\varepsilon}y_n^{(\text{IV})}. \quad (1.9)$$

§ 2. The Stability Region

For the model equation

$$y'=\lambda y, \quad (2.1)$$

the eigenequation of scheme (1.7) is

$$\Big(1-\frac{9}{11}p-\frac{9}{11}p\varepsilon\lambda\Big)\mu^3-\Big(\frac{18}{11}-\frac{12}{11}p\Big)\mu^2+\Big(\frac{9}{11}-\frac{3}{11}p\Big)\mu-\frac{2}{11}=0. \quad (2.2)$$

The stability region in the $\varepsilon\lambda$-plane is the outside part of the following curve ($\mu=e^{i\theta}$, $\theta=0°-360°$):

$$\varepsilon\lambda=\frac{\Big(1-\frac{9}{11}p\Big)\mu^3-\Big(\frac{18}{11}-\frac{12}{11}p\Big)\mu^2+\Big(\frac{9}{11}-\frac{3}{11}p\Big)\mu-\frac{2}{11}}{\frac{9}{11}p\mu^3}. \quad (2.3)$$

The outlines for $p=1.0$, 0.9, 0.8, 0.5 are given in Figs. 1 to 4, where $p=1.0$ means $h\to\infty$. Because of the symmetry we only give the upper half part.

Unfortunately, when $p=1$ the unstability region includes a section of negative real axis, and this will cause some trouble in practice. Therefore, we need to find a value p_0 so that when $p<p_0$ the stability region includes the whole negative real axis. We introduce a lemma as follows ([2]):

Lemma. The roots μ_i ($i=1, 2, 3$) of a real coefficient cubic polynomial

$$\mu^3+\tilde{p}\mu^2+\tilde{q}\mu+\tilde{r}$$

satisfy $|\mu_i|\leqslant1$, if and only if

(i) $1+\tilde{r}>0$, $1-\tilde{r}>0$;

(ii) $1+\tilde{p}+\tilde{q}+\tilde{r}>0$, $1-\tilde{p}+\tilde{q}-\tilde{r}>0$;

(iii) $1-\tilde{q}+\tilde{p}\tilde{r}-\tilde{r}^2>0$.

It is easy to check that the left-hand side of (2.2) satisfies (i) and (ii) of the lemma for any real $\varepsilon\lambda<0$ and $0<p\leqslant1$. As for (iii), we have

$$\Big(1-\frac{9}{11}p-\frac{9}{11}p\varepsilon\lambda\Big)^2(1-\tilde{q}+\tilde{p}\tilde{r}-\tilde{r}^2)$$
$$=\frac{54}{121}(1-p)^2+\frac{135p^2-117p}{121}\varepsilon\lambda+\frac{81p^2}{121}(\varepsilon\lambda)^2.$$

If $\varepsilon\lambda$ is real negative, it no longer satisfies (iii) for any $0<p\leqslant1$. But we see that, if $p\to0$, (iii) is satisfied for any $\varepsilon\lambda<0$, so there exists a value p_0, for which if $p<p_0$, (iii) holds for any $\varepsilon\lambda<0$. In order to find p_0 we regard it as a quadratic equation of $\varepsilon\lambda$, and from the discriminant we get