APPLICATION OF THE REGULARIZATION METHOD TO THE NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION (II)*

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§1

The main purpose of this paper is to use the regularization method to solve the following integral equation of the Abel type

$$A_f = 2\int_p^\infty \frac{r f(r)}{\sqrt{r^2-p^2}}\, dr = g(p), \tag{1}$$

which is of great importance in many applications[1].

Suppose that the function $f_T(r)$ having a continuous first derivative and compact support $[0, T]$ is a solution of equation (1) with right-hand side $g_T(p)$, i.e.,

$$A_{f_T} = 2\int_p^T \frac{r f(r)}{\sqrt{r^2-p^2}}\, dr = g_T(p)$$

and is yet to be found.

There are two cases to be considered:

Case I. The position of the right end point of the compact support $[0, T]$ is given exactly in advance.

Case II. The position is known only approximately.

The problem of solving Abel's integral equation

$$A_z = \int_0^x \frac{z(s)}{(x-s)^a}\, ds = u(x)$$

has been studied in [2]. In Case I in exactly the same way one can easily see that the analogous problem of determining the solution $f(r)$ of the Abel type integral equation (1) in the space $C[0, T]$ from the initial data $g(p)$ in the space $L_2[0, T]$ is not well-posed on the pair of spaces (C, L_2) ([3] p. 16 and [2]) and that the problem of constructing approximate solutions can be solved in accordance with the method described in [2].

In Case II we are thus forced to adopt a somewhat different approach to solve problem (1) for $f_T(r)$. In the following we shall treat this problem in detail.

§2

In Case II because of the ambiguity of the position of the right end point we

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prefer to study problem (1) on the pair of spaces $(\bar{C}, L_2)$, where

$$L_2 = L_2[0, \bar{T}],$$

$$\bar{C} = \bar{C}[0, \bar{T}] = \{f(r): f(r) \text{ is continuous on } [0, \bar{T}] \text{ and has compact support } [0, \xi],\ 0 < \xi \leqslant T,\ T < \bar{T}\},$$

$$\|f\|_{\bar{C}} = \max |f(r)|.$$

The problem of determining the solution $f(r)$ from the initial data $g(p)$, like the problem considered in [2], is not well-posed on $(\bar{C}, L_2)$. For, in the first place, the set $A\bar{C}$ does not coincide with L_2. Secondly, the inverse operator A^{-1} is not continuous.

Furthermore, it should be noted that the reciprocity formula for $f(r)$ holds[1]:

$$f(r) = \frac{-1}{\pi r}\frac{d}{dr}\int_r^\infty \frac{pg(p)}{\sqrt{p^2 - r^2}}\,dp.$$

Below, following[4], we shall employ the regularization method for the Abel type equation (1) to construct a regularizing operator that provides a stable method for determining approximate solutions. For this purpose we consider the functional $M^\alpha[f, g]$ defined on $\bar{C}_1[0, T]$:

$$M^\alpha[f, g] = \|Af - g\|^2_{L_2} + \alpha\int_0^T [f^2(r) + f'(r)^2]\,dr$$

$$= \int_0^{\bar{T}}\left[2\int_p^T \frac{rf(r)}{\sqrt{r^2 - p^2}}\,dr - g(p)\right]^2 dp + \alpha\int_0^{\bar{T}} [f^2(r) + f'(r)^2]\,dr,$$

$$\bar{C}_1 = \bar{C}_1[0, \bar{T}] = \{f(r): f(r) \in \bar{C},\ f(r) \text{ has a continuous derivative}\}.$$

Theorem 1. *For every function $g \in L_2$ and every positive parameter α, there exists a unique function $f_\alpha \in \bar{C}_1$ for which the functional $M^\alpha[f, g]$ attains its greatest lower bound, that is*

$$M^\alpha[f_\alpha, g] = \inf M^\alpha[f, g].$$

Proof. 1) This is a variational problem with free boundaries; the left and right end points of the unknown curve $f_\alpha(r)$ are on lines $r = 0$ and $p = 0$ respectively. Thus, we obtain after simple calculation the first variation δM^α of the functional M^α:

$$\delta M^\alpha = 4\int_0^\xi \left\{\int_0^r \frac{r}{\sqrt{r^2 - p^2}}\left[2\int_p^\xi \frac{tf(t)}{\sqrt{t^2 - p^2}}\,dt - g(p)\right]dp\right\}h(r)\,dr$$

$$+ 2\alpha\int_0^\xi [f(r) - f''(r)]h(r)\,dr + 2\alpha f'(r)h(r)\Big|_{r=0}^{r=\xi},$$

and hence the function $f_\alpha(r)$ should be determined by the Euler integro-differential equation

$$\alpha L[f] = 4\int_0^r \frac{r}{\sqrt{r^2 - p^2}}\left[\int_p^\xi \frac{tf(t)}{\sqrt{t^2 - p^2}}\,dt\right]dp - 2\int_0^r \frac{r}{\sqrt{r^2 - p^2}}\,g(p)\,dp,\quad L[f] = f'' - f \tag{2}$$

and the boundary conditions

$$f'(0) = 0,\quad f'(\xi) = 0,\quad f(\xi) = 0. \tag{3}$$

2) Under given boundary conditions (3) the associated homogeneous equation

$$\alpha L[f] = 4\int_0^r \frac{r}{\sqrt{r^2 - p^2}}\left[\int_p^\xi \frac{tf(t)}{\sqrt{t^2 - p^2}}\,dt\right]dp, \tag{4}$$