APPLICATION OF THE REGULARIZATION METHOD TO THE NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION*

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§ 1

In the present paper, we shall consider an ill-posed problem, the solution of Abel's integral equation with unbounded kernel

$$Az=\int_0^x \frac{z(s)}{(x-s)^a}\,ds=u(x), \quad (x, s)\in[0, 1]\times[0, 1], \qquad 0<a<1,\ u(0)=0, \quad (1)$$

where $u(x)$ is a know function in the space $L_2[0, 1]$ and $z(s)$ is the unknown function in the space $C[0, 1]$. This is an important problem encountered in practice ([1] and [2], Vol. I, 158—160).

It should be pointed out first of all that Abel's integral operator A in equation (1) possesses the properties:

1) The operator A is completely continuous. This is true because

$$\|u\|^2_{L_2}=\int_0^1\Big[\int_0^x \frac{z(s)}{(x-s)^a}\,ds\Big]^2 dx\leqslant\|z\|^2_C\int_0^1\Big[\int_0^x (x-s)^{-a}\,ds\Big]^2 dx=\frac{\|z\|^2_C}{(1-a)^2(3-2a)},$$

and

$$\|u(x+h)-u(x)\|^2_{L_2}=\int_0^1\Big[\int_0^{x+h}\frac{z(s)}{(x+h-s)^a}\,ds-\int_0^x\frac{z(s)}{(x-s)^a}\,ds\Big]^2 dx$$

$$\leqslant\|z\|^2_C\int_0^1\Big[\frac{x^{1-a}-(x+h)^{1-a}+2h^{1-a}}{1-a}\Big]^2 dx\to 0, \text{ as } h\to 0.$$

2) The operator A which maps $C[0, 1]$ onto $AC[0, 1]$ is one-to-one. This follows from the reciprocity formula ([2], Vol. I, 159)

$$z(s)=\frac{\sin \pi a}{\pi}\frac{d}{ds}\int_0^s\frac{u(x)}{(s-x)^{1-a}}\,dx.$$

Suppose that the element $z_T(s)\in C_1[0, 1]$ is a solution of equation (1) with right-hand member $u(x)=u_T(x)\in AC_1[0, 1]$, i.e.,

$$Az_T=u_T,$$

and requires to be found. However, in computation we often know only the approximate right-hand member $u_\delta(x)$ rather than the exact one $u_T(x)$, in such a case, we can speak only of finding an approximate solution $z_\delta(s)$ (i.e., one close to $z_T(s)$). Unfortunately the problem of determining the solution $z(s)$ of equation (1) in the space $C[0, 1]$ from the initial data $u(x)$ in the space $L_2[0, 1]$ is not well-posed on the pair of spaces (C, L_2) in the sense of Hadamard ([3], p. 16). First, it is

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obvious that the approximate solution $z_\delta(s)$ cannot be defined as the exact solution of the equation $Az=u_\delta$ with approximate right-hand member

$$u=u_\delta,$$

that is, it cannot be determined by

$$z_\delta=A^{-1}u_\delta,$$

since the approximate element u_δ may fail to belong to the set $AC[0, 1]$. Second, even if such a solution z_δ does exist, it will not possess the property of stability, since the inverse operator A^{-1} is not continuous. To see this, let us suppose that the approximate right-hand member $u_\delta(x)$ has the form

$$u_\delta(x)=u_T(x)+\delta^{\frac{1-a}{3}}\sin\frac{x}{\delta},$$

then

$$\|u_\delta(x)-u_T(x)\|_{L_2}\leqslant\delta^{\frac{1-a}{3}},$$

$$z_\delta(s)=z_T(s)+\frac{\sin\pi a}{\pi}\frac{d}{ds}\int_0^s\frac{\delta^{\frac{1-a}{3}}\sin\frac{x}{\delta}}{(s-x)^{1-a}}\,dx.$$

However, the difference between the solutions

$$\|z_\delta(s)-z_T(s)\|_c\geqslant|z_\delta(\delta^{\frac{1+2a}{3a}})-z_T(\delta^{\frac{1+2a}{3a}})|\geqslant\frac{\sin\pi a}{\pi}\frac{1}{2}\delta^{\frac{-(1-a)}{3}}$$

can be made arbitrarily large for sufficiently small values of δ. Thus, the requirements for a well-posed problem are not satisfied. Consequently, the problem (1) is ill-posed.

§ 2

A method of solving ill-posed problems, widely used in computational work is the regularization method. It consists in constructing a regularizing operator. An operator $R(u, \alpha)$ depending on a parameter α is called a regularizing operator for the equation $Az=u$ in a neighborhood of $u=u_T$ if

1) there exists a positive number δ_1 such that the operator $R(u, \alpha)$ is defined for every $\alpha>0$ and every u in $L_2[0, 1]$ for which

$$\|u-u_T\|_{L_2}\leqslant\delta\leqslant\delta_1.$$

2) there exists a function $\alpha=\alpha(\delta)$ of δ such that, for every $\varepsilon>0$, there exists a number $\delta(\varepsilon)\leqslant\delta_1$ such that the inclusion $u_\delta\in L_2[0, 1]$ and the inequality

$$\|u_\delta-u_T\|_{L_2}\leqslant\delta(\varepsilon)$$

imply

$$\|z_\alpha-z_T\|_c\leqslant\varepsilon,$$

where

$$z_\alpha=R(u_\delta, \alpha(\delta))\quad ([3], \text{ p. } 55).$$

It is obvious that every regularizing operator $R(u_\delta, \alpha(\delta))$ defines a stable method of constructing approximate solutions. Thus, the problem of finding an approximate solution reduces to

1) constructing the operator $R(u, \alpha)$, and

2) selecting the regularization parameter $\alpha=\alpha(\delta)$ from the discrepancy δ.

The regularizing operator for the Fredholm integral equation of the first kind