EXTRAPOLATION CORRECTION AND COMBINED ALGORITHMS FOR SOLVING PARABOLIC EQUATIONS BY THE DIFFERENCE METHOD*

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Introduction

The primary motivation of the work described in this paper comes from [1], in which Lin Qun and Lü Tao have successfully presented the so-called splitting extrapolation process and obtained a series of extrapolation, correction and combined algorithms for the solution of multidimensional integral equations and elliptic equations. These algorithms can give higher accuracy results and are especially suitable for parallel computation.

In this paper, the idea of [1] is generalized to evolution equations. It is proved that, under certain conditions, the finite-difference approximate solution of the differential problem can be expanded to power series of the mesh width, so that the splitting extrapolation process presented in [1] can also be used. Some correction and combined algorithms for the solution of a heat equation are given, so as to obtain higher accuracy results. In addition, the accuracy of the method by B. K. Saul'ev[2] taking the arithmetic mean of the non-symmetric schemes is discussed.

1.

In this section we shall confine ourselves to linear evolution equations. An expression connecting the finite-difference solution with the analytic one will be given as a starting point of later discussions. As the condition of our main theorem requires some smoothness for the solution of the differential problem, this paper is mainly concerned with the Cauchy problem and initial-boundary value problem with the boundary condition of the first kind for the parabolic equation. It is well known that these problems are properly posed[3]. As for the hyperbolic equation, our conclusion will also be true provided that the solution is smooth enough.

In a suitably chosen Banach space, the problem in question can be expressed, following the notation of [4], by

$$\begin{cases} \frac{d}{dt} U(t) = AU(t) + g(t), \quad 0<t<T, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where A is a linear operator and, as in [4], does not depend on t. The boundary

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conditions, if any, will be supposed to be homogeneous linear, and be contained in the definition of the domain $D(A)$.

In solving problem (1.1) by the difference method, we choose mesh widths $\Delta t = h_0$, $\Delta x_i = h_i = g_i(\Delta t)$, and suppose $g_i(\Delta t) \to 0$ as $\Delta t \to 0$, $i=1, 2, \cdots, p$. The corresponding difference scheme is denoted by

$$\begin{cases} B_1(\Delta t)u^{n+1} = B_0(\Delta t)u^n + g^n, \\ u^0 = U_0, \end{cases} \tag{1.2}$$

where u^n is the numerical approximation to $U(n\Delta t)$, $g^n = g(n\Delta t)$, and $B_0(\Delta t)$, $B_1(\Delta t)$ are linear difference operators.

The difference problem (1.2) is said to provide a consistent approximation for the initial-value problem (1.1) if, for any function $u(t, X)$ having continuous partial derivatives up to the order $m+1$, and $u(t) \in D(A)$,

$$\left\| B_1(\Delta t)\, u(t+\Delta t) - B_0(\Delta t)\, u(t) - \left\{ \frac{d}{dt} - A \right\} u(t) \right\| = \Big\| \sum_{\alpha < |\beta| < m} (R_\beta u)(t) h^\beta \Big\| + O(\tilde{h}^{m+1}), \tag{1.3}$$

where the integer $\alpha \geqslant 1$, $h = (h_0, h_1, \cdots, h_p)$, $\tilde{h} = \max_i h_i$, $\beta = (\beta_0, \beta_1, \cdots, \beta_p)$, β_i are positive integers, $|\beta| = \beta_0 + \beta_1 + \cdots + \beta_p$, $h^\beta = h_0^{\beta_0} h_1^{\beta_1} \cdots h_p^{\beta_p}$ and $(R_\beta u)(t)$ are elements in the Banach space (in fact they are the derivatives of the function $u(t, X)$).

Set $C(\Delta t) = B_1^{-1}(\Delta t) B_0(\Delta t)$. The finite-difference approximation (1.2) is said to be stable, if, for some constants $\tau > 0$ and $M > 0$,

$$\|C(\Delta t)^n\| \leqslant M, \quad 0 < \Delta t < \tau, \ 0 \leqslant n\Delta t \leqslant T. \tag{1.4}$$

In addition, it is assumed that there exists a constant $N > 0$ such that

$$\|B_1^{-1}(\Delta t)\| \leqslant N \Delta t. \tag{1.5}$$

This condition can be satisfied by usual schemes.

We can now prove

Theorem. *If*

1) *equation* (1.1) *and the finite-difference approximation are consistent in the sense of* (1.3);

2) *scheme* (1.2) *is stable; and*

3) *condition* (1.5) *is satisfied,*

then, for any solution of equation (1.1) *having continuous partial derivatives of order* $2m+1$, *the following equality holds*:

$$u^n = U(n\Delta t) + \sum_{\alpha < |\beta| < m} V_\beta(n\Delta t) h^\beta + Q(n\Delta t), \tag{1.6}$$

where $V_\beta(t)$ *are independent of* h *and* $\|Q(n\Delta t)\| = O(\tilde{h}^{m+1})$.

Proof. For any smooth solution of (1.1), from (1.3), we have

$$B_1(\Delta t)U(t+\Delta t) - B_0(\Delta t)U(t) = g(t) + \sum_{\alpha < |\beta| < m} (R_\beta U)(t) h^\beta + R_0(t), \tag{1.7}$$

where $\|R_0(t)\| = O(\tilde{h}^{m+1})$. At the mesh points $t = n\Delta t$, subtracting (1.7) from (1.2), we get

$$B_1(\Delta t)\{u^{n+1} - U(t+\Delta t)\} - B_0(\Delta t)\{u^n - U(t)\} = -\sum_{\alpha < |\beta| < m} (R_\beta U)(t) h^\beta - R_0(t). \tag{1.8}$$