THE ERROR ESTIMATES FOR CRANK-NICOLSON GALERKIN METHODS FOR QUASI-LINEAR PARABOLIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS*

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§ 1. Introduction

There have been a lot of papers on finite element analyses of the linear and nonlinear parabolic equations, but only a few are concerned with the problems in which the boundary conditions are of mixed type——the problems that are frequently encountered in engineering applications.

In [5], the author considered the semi-discrete Galerkin methods for quasi-linear parabolic equations with nonlinear third mixed boundary conditions. In this paper, we consider a discrete time Galerkin approximation for the same parabolic problem investigated in [5]. In § 2, a Crank-Nicolson Galerkin procedure for the problem is described and its solvability discussed. In § 3 and § 4, H^1-norm and L_2-norm error estimates with optimal approximating order with respect to the space mesh parameter h are developed respectively.

Consider the following parabolic equation and associated initial value and boundary conditions:

$$(\mathrm{A})\begin{cases} \dfrac{\partial u}{\partial t}=\nabla\cdot(k(x,u)\nabla u)+\boldsymbol{b}(x,u)\cdot\nabla u+f(x,t;u), \\ \qquad\qquad (x,t)\in\Omega\times(0,T], & (1.1)\\ u=0,\quad (x,t)\in\partial\Omega_1\times[0,T], \\ k(x,u)\nabla u\cdot\boldsymbol{\nu}+\sigma(x,u)u=g(x,t;u),\quad (x,t)\in\partial\Omega_2\times[0,T], & (1.2)\\ u(x,0)=u_0(x),\quad x\in\Omega, & (1.3)\end{cases}$$

where Ω is a bounded domain in R^n with piecewise smooth boundary and satisfies the cone condition, $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$, meas $(\partial\Omega_1)>0$, $\boldsymbol{b}(x,u)=(b_1(x,u), b_2(x,u),\cdots, b_n(x,u))$ and $\boldsymbol{\nu}=(\nu_1,\nu_2,\cdots,\nu_n)$ is the unit exterior normal of $\partial\Omega_2$.

Assume that k, $\boldsymbol{b}$, σ, f and g satisfy the following

Condition (A_1).

(i) There exist constants k_*, k^* such that

$$\begin{aligned}&0<k_*\leqslant k(x,p)\leqslant k^*,\ |b_i(x,p)|\leqslant k^*,\quad \forall(x,p)\in\overline{\Omega}\times R^1;\\ &0\leqslant\sigma(x,p)\leqslant k^*,\quad \forall(x,p)\in\partial\Omega_2\times R^1.\end{aligned}\tag{1.4}$$

(ii) k, b_i $(i=1,2,\cdots,n)$, f, σ, g are uniformly Lipschitz continuous with

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respect to their $(n+1)$th variable with Lipschitz constant L; for each $t\in[0, T]$, $f(x, t; 0)\in L_2(\Omega)$ and $g(x, t; 0)\in L_2(\partial\Omega_2)$; and also, f, g are continuous in variable t; $u_0(x)\in H^1_*(\Omega)$, where

$$H^1_*(\Omega)=\{v: v\in H^1(\Omega),\ v|_{\partial\Omega_1}=0\}.$$

In the above notations, $H^r(\Omega)$ are usual Hilbert-Sobolev spaces on Ω with norm $\|\cdot\|_r$, the subscript will be omitted in the case $r=0$. Analogously, let $H^r(\partial\Omega)$ denote Sobolev trace spaces on $\partial\Omega$ with norm $\|\cdot\|_{r,\partial\Omega}$; specifically, in the case $r=0$, $H^0(\partial\Omega)=L_2(\partial\Omega)$ and

$$\|v\|^2_{0,\partial\Omega}=\int_{\partial\Omega} v^2\,ds.$$

Let X be a Banach space, and $\varphi(t)$ a map $[0, T]\to X$. Define

$$\|\varphi\|_{L_p(X)}=\left(\int_0^T\|\varphi\|^p_X(t)\,dt\right)^{1/p},\ 1\leqslant p<+\infty;\quad \|\varphi\|_{L_\infty(X)}=\sup_{0<t<T}\|\varphi\|_X(t).$$

The spaces $L_p(X)$ and $L_\infty(X)$ are the set of all φ such that above norm are finite respectively.

Let J be a positive integer, and $\Delta t=T/J$ a time step. Let $t_j=j\Delta t$, and $\varphi^j=\varphi(t_j)$. Define

$$\|\varphi\|_{\tilde L_2(X)}=\left(\sum_{j=0}^J\|\varphi^j\|^2_X\Delta t\right)^{1/2},\quad \|\varphi\|_{\tilde L_2^\prime(X)}=\left(\sum_{j=0}^{J-1}\|\varphi^{j+1/2}\|^2_X\Delta t\right)^{1/2},$$

$$\|\varphi\|_{\tilde L_\infty(X)}=\max_{0<j<J}\|\varphi^j\|_X,\quad \|\varphi\|_{\tilde L_\infty^\prime(X)}=\max_{0<j<J-1}\|\varphi^{j+1/2}\|_X,$$

where

$$\varphi^{j+1/2}\equiv(\varphi(t_j)+\varphi(t_{j+1}))/2.$$

For convenience, we write $\|\varphi\|_{L_p(H^r(\Omega))}\equiv\|\varphi\|_{L_p(H^r)}$, $\|\varphi\|_{L_\infty(L_\infty(\Omega))}\equiv\|\varphi\|_{L_\infty(L_\infty)}$ and $u(t)\equiv u(X, t)$, $b_i(u)\equiv b_i(x, u)$, $f(u)\equiv f(x, t, u)$ etc.

The weak form of problem (A) is the following: find a differentiable map $u(t)$: $[0, T]\to H^1_*(\Omega)$ such that

$$\text{(B)}\quad\begin{cases}\left(\dfrac{\partial u}{\partial t}, v\right)+a(u; u, v)=(\boldsymbol{b}(u)\cdot\nabla u, v)+(f(u), v)+\langle g(u), v\rangle, \\ \qquad\qquad\qquad\qquad \forall v\in H^1_*(\Omega),\ 0<t\leqslant T, \\ u(0)=u_0,\end{cases}\tag{1.5}$$

where

$$(w, v)=\int_\Omega wv\,d\Omega,\quad \langle w, v\rangle=\int_{\partial\Omega_2} wv\,ds,$$

$$a(Q; w, v)=\int_\Omega k(Q)\nabla w\cdot\nabla v\,d\Omega+\int_{\partial\Omega_2}\sigma(Q)wv\,ds.\tag{1.6}$$

From (1.4)

$$k_*|v|^2_1\leqslant a(Q; v, v)\leqslant k^*(|v|^2_1+\|v\|^2_{0,\partial\Omega}),\quad \forall Q,\ v\in H^1_*(\Omega),\tag{1.7}$$

where the semi-norm

$$|v|^2_1=(\nabla v, \nabla v)=\sum_{i=1}^n\|v_{x_i}\|^2.$$

Throughout this paper, we shall always suppose that the solution $u(t)$ of problem (B) exists uniquely and use letters C, C_i, C_i^*, s to denote generic constants which have different values in different inequalities.