ON DISCONTINUOUS FINITE ELEMENT APPROXIMATION FOR THE SOLUTION OF TRICOMI'S PROBLEM*

HUANG MING-YOU (黄明游)
(Jilin University, Jilin, China)

§ 1. Introduction

In this paper we shall discuss a discontinuous finite element approximation for the solution of Tricomi's problem

$$\begin{cases} y\Phi_{,xx}-\Phi_{,yy}=f & \text{in } \Omega, \\ \Phi=0 & \text{on } \Gamma^2, \Gamma^3 \text{ and } \Gamma^4, \\ 2\Phi_{,x}+\Phi_{,y}=0 & \text{on } \Gamma^5, \\ \Phi & \text{unspecified on } \Gamma^1, \end{cases} \tag{1.1}$$

where Ω as shown in the figure of section 2 is a domain in the (x, y) plane bounded by the characteristics Γ^1 and Γ^2 passing through the points $(0, 1)$ and $(0, -1)$ respectively for $y\geqslant 0$ and bounded by the rectangle with sides Γ^3, Γ^4 and Γ^5 for $y<0$. This problem is a linear mixed type problem with equation hyperbolic in the part $y>0$ of Ω, elliptic in the other part $y<0$ of Ω and parabolic on the line $y=0$.

It is known that by transformation problem (1.1) can be reduced to a first order symmetric positive system. In [1] and [4] a finite element method for the solution of Tricomi's problem has been presented in the form of first order system, where the finite element space is a subspace of $H^1(\Omega)$ consisting of piecewise polynomials of degree $\leqslant r$. And an error bound in L_2 norm of $O(h^r)$ for this continuous finite element method was shown, based on the results of Lesaint in [2] for the finite element method for first order symmetric positive systems. This error estimate is not optimal compared with the approximation properties of the finite element space employed.

The discontinuous finite element approximation for Tricomi's problem to be discussed is constructed first by transforming (1.1) into a first order symmetric positive system and then applying a discontinuous finite element procedure presented in [3] to this reduced first order system. We aim at studying the stability and convergence properties of the finite element method for Tricomi's problem in a wider situation where the finite element space is only a subspace of $L_2(\Omega)$. We shall prove that the rate of convergence is $O(h^{r+\frac{1}{2}})$ provided the finite element space is chosen as the space of piecewise polynomials of degree $\leqslant r$. This result shows the effectiveness of the discontinuous finite element, and it also is an

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improvement of the known result for the continuous finite element method.

§ 2. Reduction of Tricomi's Problem

We introduce the following transformation of the independent variables

$$u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1=e^{-\lambda x}\Phi_{,x}, \quad u_2=e^{-\lambda x}\Phi_{,y}.$$

Then the equation in (1.1) can be written in form of first order system

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{,x}+\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{,y}+\begin{pmatrix} \lambda y & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} e^{-\lambda x}f \\ 0 \end{pmatrix},$$

which is symmetric. Multiplying this system by matrix

$$T=\begin{pmatrix} a & by \\ b & a \end{pmatrix}$$

we obtain

$$Lu\equiv A_1u_{,x}+A_2u_{,y}+A_3u=F, \tag{2.1a}$$

where

$$A_1=\begin{pmatrix} ay & by \\ by & a \end{pmatrix}, \; A_2=\begin{pmatrix} -by & -a \\ -a & -b \end{pmatrix}, \; A_3=\begin{pmatrix} \lambda ay & \lambda by \\ \lambda by & \lambda a \end{pmatrix}, \; F=\begin{pmatrix} ae^{-\lambda x}\cdot f \\ be^{-\lambda x}\cdot f \end{pmatrix}.$$

It can be seen that with the choice of $a=2$, $b=1$ and $\lambda=0.1$ system (2.1a) is a symmetric positive system in Ω. In the sequel we shall fix this choice. Let L^* be the formal adjoint of L

$$L^*u\equiv-(A_1u)_{,x}-(A_2u)_{,y}+A_3^Tu.$$

We have

$$L+L^*=A_3+A_3^T-\frac{\partial A_1}{\partial x}-\frac{\partial A_2}{\partial y}=\begin{pmatrix} 0.4y+1 & 0.2y \\ 0.2y & 0.4 \end{pmatrix},$$

and $(L+L^*)$ is strictly positive, i.e. there exists a constant $\alpha>0$ such that

$$u^T(L+L^*)u\geqslant\alpha u^T\cdot u \quad \text{in } \Omega \text{ for } u\in R^2. \tag{2.2}$$

We now examine the corresponding boundary condition of system (2.1a). The boundary $\partial\Omega$ of Ω is shown in the figure. Let $n=(n_x, n_y)$ be the outer unit normal on $\partial\Omega$.

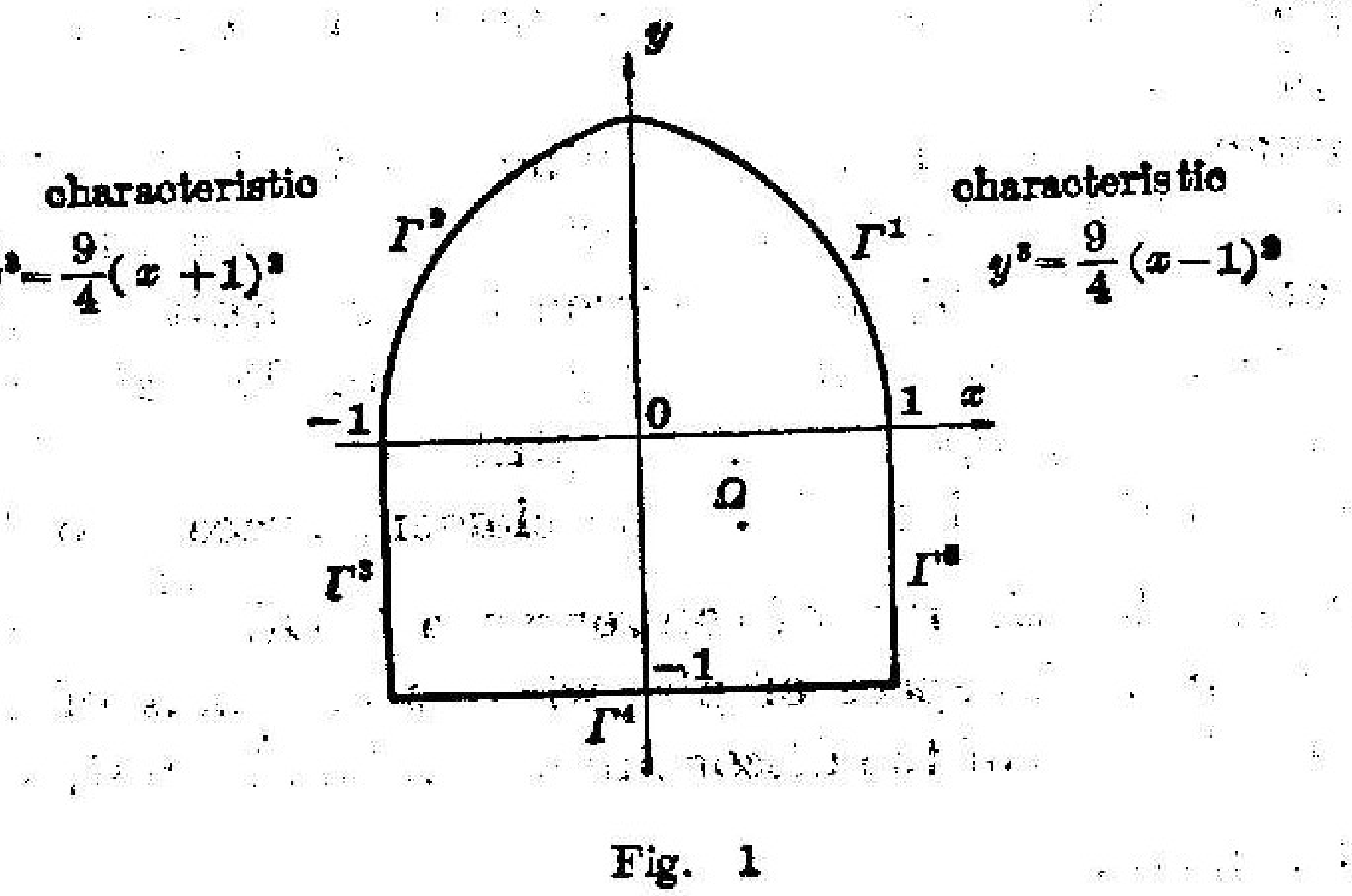

Fig. 1