INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 1, Number 1, Pages 25–47

A SURVEY ON MULTIPLE LEVEL SET METHODS WITH APPLICATIONS FOR IDENTIFYING PIECEWISE CONSTANT FUNCTIONS

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Abstract. We try to give a brief survey about using multiple level set methods for identifying piecewise constant or piecewise smooth functions. A general framework is presented. Application using this general framework for different practical problems are shown. We try to show some details in applying the general approach for applications to: image segmentation, optimal shape design, elliptic inverse coefficient identification, electricall impedance tomography and positron emission tomography. Numerical experiments are also presented for some of the problems.

Key Words. survey, level set methods, image segmentation, inverse problems, optimal shape design, electrical impedance tomography, positron emission tomography.

1. Introduction

In this work, we are trying to give a brief survey about using multiple level set methods for identifying piecewise constant or piecewise smooth functions. We try to present a general framework and then show various applications of this basic approach. The general approach presented here is originated from [12, 55, 8]. The applications we have surveyed have used this general approach or could be reformulated using this general approach for multiple level set ideas.

The general minimization problem we shall consider in this work is given in the form:

$$\min_{q\in K} F(q), \tag{1}$$

where K is a space or set containing piecewise constant functions over a given domain Ω and possibly with some other extra constraints. Such kinds of minimization problems arise from inverse problems, optimal shape design problems, medical imaging and other applications.

In order to find a piecewise constant function, we essentially need to find the values for the constants and the location of the discontinuities. For some applications, the values of the constants are known and we only need to find the locations of the discontinuities. For two-dimensional problems, to find the locations of the discontinuities is to find the curves that separate the constant regions. For three dimensional problems, to find the discontinuity is to find the surface between the

Received by the editors November 10, 2003.
2000 *Mathematics Subject Classification.* 49Q10, 49L99, 35C44, 35R30, 65J20.
We acknowledge support from the Norwegian Research Council (project numbers 135420/431), from the NSF under contracts NSF ACI-0072112, NSF INT-0072863 and NSF DMS 99-73341, from the ONR under grant N00014-96-1-10277, and from the NIH under grant P20MH65166.

regions. In practical simulations, we need to use a mesh or grid. For the applications we shall consider, each constant region normally contains many mesh or grid points.

Minimization using shape derivatives have been used for finding curves and surfaces. We shall give a brief overview in Section 2 for this kind of approach. One of the potential limitations of this kind of approach is that it is difficult to handle the case that the curve or surface may disappear, merge with each other, or pinch off with each other. In this work, we shall present a different approach of using level sets to represent piecewise constant functions and embed this representation in frameworks for solving a variety of inverse problems and minimization problems with piecewise constant functions. As computational techniques, level set methods have several advantages in moving curves in 2D and surfaces in 3D. In section 3, we give an overview of the multiple level set idea first proposed in [55]. We try to supply some of the details in calculations related to gradient methods for level set ideas. It is shown that we can easily combine level set methods to calculate gradients for the considered minimization problems. Sections 4-9 are devoted to different applications. In §4, we show how to apply the level set methods for segmentation of digital images following [11, 12, 10, 9, 55]. The minimization functional is slightly different from the original Chan-Vese functional [9, 55]. In §5, we reformulate the optimal shape design problem of Osher and Santosa [42] into the framework of multiple level set methods. Applications for identifying the coefficient from an elliptic equation are presented in §6 following Chan and Tai [8]. In the original calculation of [8], augmented Lagrangian method was used to deal with the equation constraint. In §6, we show the formulation without using the augmented Lagrangian methods and some numerical tests are presented. In §7, we show applications of the level set methods for electrical impedance following Chung-Chan-Tai [18]. A similar application using level set methods for PET medical imaging is discussed in §8 following Lysaker et al [36]. In the last section §9, we show a variant of the level set methods which can be used to trace free boundary for the obstacle problems [37]. In the conclusion, we briefly mention some of the key issues in using level set methods for practical applications.

As the application of level set methods for identifying piecewise constant functions is relatively new, the works we have surveyed are mostly recent works of our research group and we must apologize for possible omissions for other recently related works.

2. Minimization using shape derivatives

If the constant values for a 2D piecewise constant function are known, then we just need to identify the location of the discontinuities. For such a kind of applications, the minimization problem (1) can often be transformed into the following minimization problem:

$$\min_{\Gamma} F(\Gamma). \tag{2}$$

That is, we trying to find a curve Γ which minimizes the functional $F(\Gamma)$. Traditionally, a curve is parameterized as:

$$x = x(s) \quad s \in [0,1].$$

Correspondingly, the minimization problem (2) could be transformed into

$$\min_{x(s)} F(x(s)). \tag{3}$$