A FAMILY OF PARALLEL AND INTERVAL ITERATIONS FOR FINDING ALL ROOTS OF A POLYNOMIAL SIMULTANEOUSLY WITH RAPID CONVERGENCE*

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Abstract

This paper suggests a family of parallel iterations with parameter p ($p=1, 2, \cdots$) for finding all roots of a polynomial simultaneously. The convergence of the methods is of order $p+2$. The methods may also be applyed to interval iterations.

Let $x\in\mathbb{C}$,

$$\Delta_0(x)=\Delta_0[f;x]=1,$$

$$\Delta_p(x)=\Delta_p[f;x]=\begin{vmatrix}\sigma_1 & 1 & & 0\\ \sigma_2 & \sigma_1 & 1 & \\ \vdots & \ddots & \ddots & \ddots \\ \sigma_{p-1} & \cdots & \sigma_2 & \sigma_1 & 1\\ \sigma_p & \cdots & \sigma_3 & \sigma_2 & \sigma_1\end{vmatrix},$$

where

$$\sigma_\nu=\sigma_\nu(x)=\frac{f^{(\nu)}(x)}{\nu!f(x)}.$$

Expanding the determinant in the first column, we see that $\Delta_p(x)$ satisfies the following recursion relations:

$$\Delta_p(x)=\sum_{\nu=1}^{p}(-1)^{\nu+1}\sigma_\nu(x)\Delta_{p-\nu}(x),$$

which are called first recursion relations. To find a zero of equation

$$f(x)=0,$$

a family of iteration method

$$x^{(n+1)}=x^{(n)}-\frac{\Delta_{p-1}[f;x^{(n)}]}{\Delta_p[f;x^{(n)}]},\quad n\in\mathbb{N}_0$$

has been discussed in [1], where p is any positive integer. The method is of order $p+1$ if the zero is simple. The special cases of the method contain the well-known Newton iteration ($p=1$) and Halley iteration ($p=2$). From the first recursion relations we see that the method is connected with Bernoulli's method for finding a root of a polynomial. On the other hand, if we write $\sigma_\nu=\dfrac{f^{(\nu+1)}(x)}{\nu!f'(x)}$, the method becomes immediately an optimum seeking method for finding a minimizer of $f(x)$, which is the same as the method of Hua and Xia[2] in evaluation and rates of convergence. Obviously, the method may be applied to the operator in the Banach

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space provided a suitable arrangement for the computed pattern is made.

In this paper we will transform the family of the methods into another family, which can help to find simultaneously all zeros of a polynomial, while its order of convergence is raised by 1. Moreover, we show that the family is suitable especially for parallel and interval operations.

First, it is clear that the generating function of $\Delta_p(x)$ is $\frac{f(x)}{f(x-z)}$, that is,

$$\Delta_p(x)=\frac{\partial^p}{p!\partial z^p}\frac{f(x)}{f(x-z)}\bigg|_{z=0}.$$

Using Leibniz' formula for

$$\frac{\partial}{\partial z}\frac{f(x)}{f(x-z)}=\frac{f(x)}{f(x-z)}\cdot\frac{f'(x-z)}{f(x-z)},$$

we obtain

$$\frac{\partial^p}{p!\partial z^p}\frac{f(x)}{f(x-z)}$$

$$=\frac{1}{p}\sum_{\nu=1}^{p}\left(\frac{\partial^{\nu-1}}{(\nu-1)!\partial z^{\nu-1}}\frac{f'(x-z)}{f(x-z)}\right)\left(\frac{\partial^{p-\nu}}{(p-\nu)!\partial z^{p-\nu}}\frac{f(x)}{f(x-z)}\right).$$

Let $z=0$. We have the recursion relations

$$\Delta_p(x)=\frac{1}{p}\sum_{\nu=1}^{p}s_\nu(x)\Delta_{p-\nu}(x),$$

where $$s_\nu(x)=\frac{\partial^{\nu-1}}{(\nu-1)!\partial z^{\nu-1}}\frac{f'(x-z)}{f(x-z)}\bigg|_{z=0}=\frac{(-1)^{\nu-1}}{(\nu-1)!}\left(\frac{f'(x)}{f(x)}\right)^{(\nu-1)},$$

which are called the second recursion relations.

If $f(x)$ is a polynomial of degree N, we have

$$\frac{f'(x)}{f(x)}=\sum_{i=1}^{N}\frac{1}{x-\xi_i},$$

where $\xi_1, \cdots, \xi_N$ are all zeros of $f(x)$. From above, it is seen that $s_\nu(x)$ is the sum of the terms of power ν of $\frac{1}{x-\xi_i}$:

$$s_\nu=\sum_{i=1}^{N}\frac{1}{(x-\xi_i)^\nu}.$$

Also from $$\frac{f(x)}{f'(x-z)}=\prod_{i=1}^{N}\frac{1}{1-\frac{z}{x-\xi_i}},$$

it is seen that $\Delta_p(x)$ is the sum of all homogeneous products of degree p of $\frac{1}{x-\xi_i}$. We denote it by Bell's polynomials $Y_p(z_1, \cdots, z_p)$ in terms of $s_\nu=s_\nu(x)$ as follows (see, e. g., [3], 74—84):

$$\Delta_p(x)=\frac{1}{p!}Y_p(s_1, s_2, 2!s_3, \cdots, (p-1)!s_p)\overset{\text{df}}{=}B_p(s_1, s_2, \cdots, s_p).$$

Using the second recursion relations of $\Delta_p(x)$ or Bell's polynomials, the expressions of

$$B_p=B_p(s_1, s_2, \cdots, s_p)$$

for $p=1, 2, \cdots, 8$ are as follows:

$$B_1=s_1;$$

$$B_2=\frac{1}{2}s_2+\frac{1}{2}s_1^2,$$