EXISTENCE OF RATIONAL INTERPOLATION FUNCTION AND AN OPEN PROBLEM OF P. TURÁN*

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Abstract

An existence theorem of rational interpolation function for the sufficient condition has correctly been stated by Macon-Dupree in [2], but some arguments in their proof are not true. In this paper:(i) A related theorem for both the sufficient and necessary condition is asserted and proved by a new and rigorous approach, namely by introducing the notion of (m/n) quasi-rational interpolant of a given function. (ii) With use of these results thus obtained an open problem proposed by P. Turán in [4] is completely solved.

§ 1. Introduction

Let $f(x)$ be a bounded real-valued function defined on an interval $[a, b]$, let m and n be non-negative integers, and let $x_i \in [a, b]$ $(i=0, 1, \cdots, m+n)$ be distinct points. The problem of rational interpolation is that of finding a rational function $R(x)=R\in \boldsymbol{R}(m, n)$ satisfying

$$R(x_i)=y_i \qquad (y_i=f(x_i);\ i=0, 1, \cdots, m+n), \tag{1.1}$$

where $\boldsymbol{R}(m, n)=\{R\colon R=N/D,\ N\in H_m,\ D\in H_n\setminus\{0\}\}$,

herein H_k denotes the class of all polynomials of degree at most k.

As we know, while the problem of polynomial interpolation is constantly solvable, the solution of the problem (1.1) does not always exist ([1, p. 2], [2, p. 754]). In order to get its possible solution, one may consider the linearized interpolation problem satisfying, instead of condition (1.1), the following linear equations:

$$N(x_i)-y_iD(x_i)=0, \qquad i=0, 1, \cdots, m+n. \tag{1.1a}$$

Now, this system of $m+n+1$ homogeneous equations in $m+n+2$ unknowns has always nontrivial solutions. However, the two problems (1.1) and (1.1a) are not equivalent. From the following known theorems one may then find their conditioned connections.

Theorem 1.1([1, p. 5], [2, p. 754], [5, p. 54]). *There exists a rational function $R\in \boldsymbol{R}(m, n)$ satisfying condition* (1.1) *if and only if the pair $\tilde{N}$ and $\tilde{D}$, obtained by dividing out all common factors in any nontrivial solution $N\in H_m$ and $D\in H_n$ of* (1.1a), *remains to be a solution of* (1.1a).

Another more practical and useful theorem may be stated in a convenient form by introducing notations for the matrices

$$C(\mu, \nu)=\begin{pmatrix} 1 & x_0 & \cdots & x_0^\mu & y_0 & y_0x_0 & \cdots & y_0x_0^\nu \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{m+n} & \cdots & x_{m+n}^\mu & y_{m+n} & y_{m+n}x_{m+n} & \cdots & y_{m+n}x_{m+n}^\nu \end{pmatrix} \tag{1.2}$$

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and $C_i(\mu, \nu)$ which denotes the matrix obtained by deleting i–th row of $C(\mu, \nu)$. We note immediately that the rational function $R(x)=\sum_{i=0}^{\mu} a_i x^i / \sum_{i=0}^{\nu} b_i x^i$, which corresponds to the nontrivial solution $\xi=(a_0 \cdots a_\mu - b_0 \cdots - b_\nu)^T$ of the equation $C(\mu, \nu)\ \xi=0$, satisfies (1.1a). Thereby, we have

Theorem 1.2([1, p. 14], [2, p. 758]). *If the rank of $C_i\ (m-1, n-1)$ is constant for $i=0, 1, \cdots, m+n$, there exists a rational function $R \in \boldsymbol{R}\ (m, n)$ satisfying the interpolation condition* (1.1).

Nevertheless, we should remark in passing that Theorem 1.2 is a true one, but within its proof, as given in the quoted references, some assertion concerning the magnitude of the rank of the related matrices can not hold. A simple counter-example, such as, given $m=n=2$, $N(x)=D(x)=x(\alpha x+\beta)$, $\beta \neq 0$, with the interpolating points (0, 0), (1, 1), (2, 1), (3, 1) and (4, 1), will do the illustration.

In the next section we make further effort to scrutinize the existence problem of rational interpolation functions. A new, rigorous proof is given for the existence Theorem 1.2 in an extended sense, namely, the condition therein given is not only sufficient but also necessary (see Theorems 2.2—2.4).

Using these concerned results as background we have solved an open problem in the field of approximation theory proposed by P. Turán in 1974 ([4, p. 79], Problem LXXXII). The problem is as follows:

Let m, n be given. For $m+n+1$ variable knots $x_0, x_1, \cdots, x_{m+n}$, what is the maximal number $M=M(m, n)$ such that, at least M of the relations (1.1) can be satisfied for any choice of y_i?

§ 2. Existence of Rational Interpolation Functions

Let $\quad \boldsymbol{R}_0(m, n)=\{R: R \in \boldsymbol{R}(m, n)$ satisfying (1.1a)$\}$.

From (1.1a), we easily get the following

Lemma 2.1. *Let $N/D \in \boldsymbol{R}_0(m, n)$. Then $N=0$ if and only if at least $m+1$ of the y_i's vanish.*

On account of Lemma 2.1, it follows that, in case $k\,(k \geqslant m+1)$ of the y_i's vanish, the problem (1.1) is solvable if and only if all of the y_i's vanish. Hence, unless specially remarked, we always assume, throughout this section, that no more than m of the y_i's can be zeros. Thus, for any $N/D \in \boldsymbol{R}_0(m, n)$, we have $N \neq 0$. Now, if we define

$$
\begin{aligned}
m^* &= \min\{\partial(N):\ N/D \in \boldsymbol{R}_0(m, n)\}, \\
n^* &= \min\{\partial(D):\ N/D \in \boldsymbol{R}_0(m, n)\},
\end{aligned}
\tag{2.1}
$$

then $m^* \geqslant 0$, $n^* \geqslant 0$. Here $\partial(P)$ denotes the degree of polynomial P, and we define $\partial(0) = -1$.

Lemma 2.2(cf. [3, p. 295]). *Let N/D and $N_1/D_1 \in \boldsymbol{R}_0(m, n)$. Then $ND_1=N_1D$.*

Lemma 2.3. *For m^* and n^* defined by* (2.1), *there exists the unique (without counting the common constant factor in numerator and denominator) $R^*=N^*/D^* \in \boldsymbol{R}_0(m, n)$ such that*

$$\partial(N^*)=m^*, \qquad \partial(D^*)=n^*;$$

and that any $R=N/D \in \boldsymbol{R}_0(m, n)$ can be reduced into R^* by dividing out some common