A NEW ITERATIVE PROCEDURE FOR THE MISSING-VALUE PROBLEM*

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Abstract

For a missing-value problem in linear models, all the iterative procedures employed up to now are entirely within the framework of the EM algorithm. This paper proposes a new iterative procedure which is not the EM algorithm in the most general case. In the procedure nothing is assumed about the error distributions in the proof of its convergence. The convergence rate is also obtained.

1. Introduction and Notations

Let us consider the linear model

$$y_i = x_i'\beta + e_i, \quad i=1, 2, \cdots, n, \tag{1.1}$$

where y_i are observations of a dependent variable y, $x_i'=(x_{i1}, \cdots, x_{ip})$ are values of p independent variables, β is an unknown p-vector, and e_i are random errors with

$$E(e_i)=0, \quad \operatorname{cov}(e_i, e_j)=\begin{cases}\sigma^2, & i=j,\\ 0, & i\neq j.\end{cases} \tag{1.2}$$

Suppose that only n_1 of the n intended observations y_i are available, while the other $n-n_1$ of y_i are missing. Without loss of generality, we may take these missing values to be the last $n-n_1$ components of the observation vector $y'=(y_1, \cdots, y_n)$. In general, $x_{n_1+1}, \cdots, x_n$, corresponding to these missing y-values, may or may not lie in $\mu(x_1, \cdots, x_{n_1})$, the space generated by the vectors $x_1, \cdots, x_{n_1}$. Under this circumstance the model (1.1) can be rewritten in the matrix notation

$$y_1 = X_1\beta + e_1, \tag{1.3}$$

$$y_2 = X_2\beta + e_2, \tag{1.4}$$

$$y_3 = X_3\beta + e_3, \tag{1.5}$$

where

$$\mu(X_2') \subset \mu(X_1'), \tag{1.6}$$

$$\mu(X_3') \cap \mu(X_1') = \{0\}, \tag{1.7}$$

and y_i are $n_i\times 1$ vectors, X_i are $n_i\times p$ design matrices, and e_i are $n_i\times 1$ error vectors.

Suppose that y_2 and y_3 are missing. Then, the sub-model (1.3) does not have the advantages of balance properties of the full-model (1.3)—(1.5), and thus its statistical computations are often complicated. It is therefore worthwhile to investigate whether or not we can retain the balance structure of the full-model (1.3)—

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(1.5) so as to analyse the sub-model (1.3). All the iterative procedures employed up to now are entirely within the framework of the EM algorithm[1-4]. The purpose of this paper is to propose a new iterative procedure for the most general case (1.3)—(1.5). Following the introduction of the procedure in Section 2, Section 3 is devoted to the proof of its convergence. Included in Section 4 is a brief discussion on the comparison between our procedure and the EM algorithm. The merits of the new procedure are that except for (1.2), nothing is assumed about the error distributions in proving its convergence, and the convergence rate is also obtained. On the contrary, no rate has been obtained for the EM algorithm.

The notations used throughout are as follows: A^- is any g-inverse of A. $\mu(A)$ is the space spaned by the columns of A. $X'=(X_1':X_2':X_3')$, $M_{ij}=X_i(X'X)^-X_j'$, $N_{ij}=X_i(X_1'X_1)^-X_j'$, $i, j=1, 2, 3$, and $\hat\beta=(X_1'X_1)^-X_1'y_1$ is the least squares solution of β in the sub-model (1.3).

2. The Iterative Procedure

Our aim is to compute $c'\hat\beta$, $c\in\mu(X_1')$, by using $(X'X)^-$ rather than $(X_1'X_1)^-$. The iterative procedure we propose is as follows.

Given a starting point $\tilde\beta^{(0)}$, solve the equation

$$X'X\beta=X_1'y_1+X_2'X_2\tilde\beta^{(0)}. \tag{2.1}$$

Denote by $\tilde\beta^{(1)}$ any special solution to (2.1). In general, let $\tilde\beta^{(k-1)}$ be the current value of β after $k-1$ cycles. Solve the equation

$$X'X\beta=X_1'y_1+X_2'X_2\tilde\beta^{(k-1)}, \tag{2.2}$$

and denote by $\tilde\beta^{(k)}$ any special solution. In this way we obtain an estimate sequence $\{\tilde\beta^{(k)}\}$.

Observe that the full-model being considered involves equation (1.5); therefore it is always possible to augment X_3 such that $X'X$ is nonsingular, or even $X'X=I_p$ in some cases. But in our proof of the convergence theorem, it is not necessary to assume $X'X$ to be nonsingular. By definition,

$$\tilde\beta^{(k)}=(X'X)^-(X_1'y_1+X_2'X_2\tilde\beta^{(k-1)}), \tag{2.3}$$

where $(X'X)^-$ is an arbitrary g-inverse of $X'X$. We will specify an arbitrary one, and denote it by $(X'X)_1^-$ for all k in (2.3).

3. Convergence of the Iteration

Theorem. *For any* $\tilde\beta^{(0)}$,

(1) $\tilde\beta^{(k)}$ *is a convergent sequence;*

(2) *denote* $\lim\limits_{k\to+\infty}\tilde\beta^{(k)}=\beta^*$; *then* $c'\beta^*=c'\hat\beta$ *for all* $c\in\mu(X_1')$;

(3) $|c'\tilde\beta^{(k)}-c'\hat\beta|=O(\max\limits_{1<i<n_2}(\alpha_i/1+\alpha_i)^{k-1})$,

where $\alpha_i\geq0$ *are the eigenvalues of* N_{22}, *defined at the end of Section* 1.

Proof. By (1.6) and (1.7), we get

$$M_{ij}=\begin{cases}X_i(X_1'X_1+X_2'X_2)^-X_j', & \text{for } i, j=1, 2, \qquad (3.1)\\ 0, & \text{for } i=1, 2, j=3, \text{ or } i=3, j=1, 2. \qquad (3.2)\end{cases}$$