THE FINITE DIFFERENCE METHOD FOR THE PERIODIC BOUNDARY AND INITIAL VALUE PROBLEM OF A CLASS OF SYSTEM OF GENERALIZED ZAKHAROV EQUATIONS*1)

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Abstract

This paper is intended to study the finite difference method for the periodic boundary and initial value problem of a class of system of generalized Zakharov equations.

1. Introduction

We consider the system of generalized Zakharov equations

$$i\varepsilon_t+\varepsilon_{xx}-\alpha(x)\eta\varepsilon+\beta(x)|\varepsilon|^2\varepsilon=0, \tag{1.1}$$

$$\eta_{tt}-\eta_{xx}+\gamma\eta_t=|\varepsilon|^2_{xx}, \tag{1.2}$$

with the periodic boundary conditions

$$\varepsilon(x,\ t)=\varepsilon(x+D,\ t),\ \eta(x,\ t)=\eta(x+D,\ t),\quad \forall x,\ t\geqslant 0, \tag{1.3}$$

and the initial conditions

$$\varepsilon(x,\ 0)=\varepsilon_0(x),\ \eta(x,\ 0)=\eta_0(x),\ \eta_t(x,\ 0)=\eta_1(x),\quad 0\leqslant x\leqslant D, \tag{1.4}$$

where $i=\sqrt{-1}$, $D>0$, γ is a real constant, $\alpha(x)$ and $\beta(x)$ are known functions which possess the period D, $\eta(x,t)$ is an unknown real function, $\varepsilon(x,t)=(\varepsilon_1(x,t),\cdots,\varepsilon_M(x,t))^T$ is an M-dimensional vector of complex function. When $M=1$, $\alpha(x)=1$, $\beta(x)=0$, and $\gamma=0$, the equation is the model equation presented by Zakharov[1,2], Morales and Lee[3,4] solved this equation by a finite difference method and obtained many results. Guo Bo-ling [5,6] proved the existence and uniqueness of the classical solution, and the stability of the difference scheme corresponding to the model equation. In this paper the existence, uniqueness of the difference solution and the convergence, stability of the scheme are proved theoretically.

2. Symbol and Convention. Difference Scheme

Let Q denote the rectangular region $[0,\ D]\times[0,\ T]$. Let Q_{hk} be a rectangular lattice covering Q, determined by the intersection of the coordinate lines

$$x=jh,\quad j=0,\ 1,\ \cdots,\ J,$$

and

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$$t=nk, \quad n=0, 1, \cdots, N,$$

where $h=D/J$ and $k=T/N$. Let x_j denote jh and t_n denote nk. We shall be concerned with functions defined on the lattice region Q_{hk}. For their forward and backward difference quotients we shall employ the following notations

$$\phi_x(x, t)=\frac{1}{h}[\phi(x+h, t)-\phi(x, t)]=\frac{1}{h}\Delta_+\phi,$$

$$\phi_{\bar{x}}(x, t)=\frac{1}{h}[\phi(x, t)-\phi(x-h, t)]=\frac{1}{h}\Delta_-\phi,$$

$$\phi_t(x, t)=\frac{1}{k}[\phi(x, t+k)-\phi(x, t)],$$

$$\phi_{\bar{t}}(x, t)=\frac{1}{k}[\phi(x, t)-\phi(x, t-k)].$$

We also introduce inner product and norms appropriate to functions defined on the lattice Q_{hk}: $(f, g)_h=\sum_{j=1}^{J} f(x_j)\overline{g(x_j)}h$, $\|f\|_h^2=(f, f)_h$, $\|f\|_{l,h}^2=\|f\|_h^2+\sum_{1\le s\le l}\left\|\frac{1}{h^s}\Delta_+^s f\right\|_h^2$, $\|f\|_{L_\infty}=\sup_{0\le j\le J}|f(x_j)|$, $\left\|\frac{1}{h^l}\Delta_+^l f\right\|_{L_\infty}=\sup_{0\le j\le J}\left|\frac{1}{h^l}\Delta_+^l f(x_j)\right|$. The norm corresponding to the space of square integrable functions is

$$\|f\|_{L_2}^2=\int_0^D |f(x)|^2 dx,$$

where f is a vector valued function. Corresponding to (1.1)—(1.4), we establish the following difference scheme, denoted by $(2)^h$ or $(2)^{h'}$,

$$(2)^h \quad \begin{cases} is_{\bar{t}}^h+s_{x\bar{x}}^h-\alpha(x)\eta^h s^h+\beta(x)|s^h|^2 s^h=0, & (2.1)\\ \eta_{t\bar{t}}^h-\eta_{x\bar{x}}^h+\gamma\eta_{\bar{t}}^h=|s^h|_{x\bar{x}}^2, & (2.2)\\ s^h(x_j, t)=s^h(x_{j+J}, t),\ \eta^h(x_j, t)=\eta^h(x_{j+J}, t), \quad \forall j,\ t\ge 0, & (2.3)\\ s^h(x_j, 0)=s_0(x_j),\ \eta^h(x_j, 0)=\eta_0(x_j),\ \eta_{\bar{t}}^h(x_j, 0)=\eta_1(x_j) & (2.4)\end{cases}$$

or

$$(2)^{h'} \quad \begin{cases} is_j^{n+1}=is_j^n-\frac{k}{h}\Delta_+\Delta_- s_j^{n+1}+k\alpha_j\eta_j^{n+1}s_j^{n+1}-k\beta_j|s_j^{n+1}|^2 s_j^{n+1}, & (2.5)\\ \eta_j^{n+1}=2\eta_j^n-\eta_j^{n-1}+\frac{k^2}{h^2}\Delta_+\Delta_-\eta_j^{n+1}-k\gamma(\eta_j^{n+1}-\eta_j^n)+\frac{k^2}{h^2}\Delta_+\Delta_-|s_j^{n+1}|^2, & (2.6)\\ s_j^n=s_{j+J}^n,\ \eta_j^n=\eta_{j+J}^n, \quad \forall j,\ n\ge 0, & (2.7)\\ s_j^0=s_0(x_j),\ \eta_j^0=\eta_0(x_j),\ \eta_j^0-\eta_j^{-1}=k\eta_1(x_j). & (2.8)\end{cases}$$

3. Existence and Uniqueness of Difference Solution

We can regard the problem $(2)^{h'}$ as a nonlinear system of unknowns s_j^{n+1} and $\eta_j^{n+1}(j=1, \cdots, J)$, but s_j^n and η_j^n, $\eta_j^{n-1}(j=1, \cdots, J)$ are known. We shall prove the existence and uniqueness of the solution for the system $(2)^{h'}$.

Lemma 1. *For two arbitrary vectors $\{u_j\}$ and $\{v_j\}$ $(j=1, \cdots, J)$, the identity*

$$\sum_{j=1}^{J} u_j\Delta_+\Delta_- v_j=-\sum_{j=1}^{J}\Delta_+u_j\Delta_+v_j-u_1(v_1-v_0)+u_{J+1}(v_{J+1}-v_J)$$

holds.

Theorem 1. *Suppose that one of the following conditions are satisfied*