THE SPLITTING EXTRAPOLATION METHOD FOR MULTIDIMENSIONAL PROBLEMS*

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Abstract

This note presents a splitting extrapolation process, which uses successively one-dimensional extrapolation procedure along only one variable with other variables kept fixed. This splitting technique is applied to the numerical cubature of multiple integrals, multidimensional integral equations and the difference method for solving the Poisson equation. For each case, the corresponding error estimates are given. They show the advantage of this method over the isotropic extrapolation along all the variables.

1. Introduction

The extrapolation method is a simple and effective numerical method for computing integration and solving differential equations in the case of one dimension. For the multidimensional problems one can use extrapolation process along all variables homogeneously, but the effort will be high. This note presents the so called splitting extrapolation process, which uses the one-dimensional extrapolation process along only one variable, the other variables fixed. We hope this method is appropriate for the parallel algorithm and will save computational effort in comparison with the isotropic extrapolation.

2. Multiple Integrals

We are concerned with the s-dimensional integral in a cube:

$$I=\int_V f(x)dx \quad \text{with} \quad V=[-1,1]^s.$$

Let us divide V in cuboids of length $h=(h_1, \cdots, h_s)$:

$$V=\bigcup_{j=1}^{n} V_j,\quad V_j=\prod_{i=1}^{s}\left[M_{ji}-\frac{h_i}{2},\ M_{ji}+\frac{h_i}{2}\right],$$

where $M_j=(M_{j1}, \cdots, M_{js})$ is the center of V_j. We define the rectangular cubature by

$$I_R(h_1, \cdots, h_s)=\sum_{j=1}^{n} \text{meas }(V_j)f(M_j)$$

and the trapezoidal cubature by

$$I_T(h_1, \cdots, h_s)=\sum_{j=1}^{n}\frac{1}{2s}\sum_{i=1}^{s}\text{meas}(V_j)\left(f(N_{ji}^{+})+f(N_{ji}^{-})\right),$$

* Received August 10, 1982.

where $N_{ji}^{\pm}=(M_{j1}, \cdots, M_{ji}\pm\frac{h_i}{2}, \cdots, M_{js})$ are the centers on the faces of V_j.

The principle of splitting extrapolation bases on the following asymptotic expansion.

Theorem 1. *For $f\in C^{(2m+2)}(V)$ we have*

$$I-I_R(h_1, \cdots, h_s)=\sum_{\substack{|p|=k\\1\leqslant k\leqslant m}} c_{2p}h^{2p}+O(h_0^{2m+2}) \tag{1}$$

with the coefficients c_{2p} independent of h and

$$p=(p_1, \cdots, p_s),\ |p|=p_1+\cdots+p_s,\ h^q=h_1^{q_1}\cdots h_s^{q_s},\ h_0=\max\{h_1, \cdots, h_s\}.$$

Proof. Note first that

$$I-I_R(h_1, \cdots, h_s)=\sum_{j=1}^{n}\int_{V_j}(f(x)-f(M_j))dx. \tag{2}$$

Then insertion of the Taylor expansion

$$f(x)-f(M_j)=\sum_{\substack{|p|=k\\1\leqslant k\leqslant 2m+1}}\frac{1}{k!}f^{(p)}(M_j)(x-M_j)^p+O(h_0^{2m+2})$$

into (2) and use of

$$\int_{V_j}(x-M_j)^q dx=\begin{cases}0 \text{ if } q \text{ contains an odd component } q_i,\\ \dfrac{\operatorname{meas}(V_j)}{\prod_{i=1}^{s}(1+2p_i)}\left(\dfrac{h}{2}\right)^{2p} \text{ when } q=2p\end{cases}$$

imply

$$I-I_R(h_1, \cdots, h_s)=\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\leqslant k\leqslant m}}\frac{f^{(2p)}(M_j)}{(2k)!}\frac{\operatorname{meas}(V_j)}{\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+O(h_0^{2m+2}).$$

By induction we assume that (1) holds for $k\leqslant m$ and come to prove it for $m+1$. In fact for $f\in C^{(2m+4)}(V)$,

$$\begin{aligned}
I-I_R(h_1, \cdots, h_s)&=\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\leqslant k\leqslant m+1}}\frac{f^{(2p)}(M_j)}{(2k)!}\frac{\operatorname{meas}(V_j)}{\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+O(h_0^{2m+4})\\
&=\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\leqslant k\leqslant m+1}}\int_{V_j}f^{(2p)}(x)dx\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}\\
&\quad-\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\leqslant k\leqslant m+1}}\int_{V_j}(f^{(2p)}(x)-f^{(2p)}(M_j))\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+O(h_0^{2m+4})\\
&=\sum_{\substack{|p|=k\\1\leqslant k\leqslant m+1}}\int_{V}f^{(2p)}(x)dx\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}\\
&\quad-\sum_{\substack{|p|=k\\1\leqslant k\leqslant m+1}}\sum_{j=1}^{n}\int_{V_j}(f^{(2p)}(x)-f^{(2p)}(M_j))dx\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+O(h_0^{2m+4}).
\end{aligned}$$

Then substitution of the asymptotic expansion (1) for $k\leqslant m$ into