NUMERICAL SOLUTIONS OF HARMONIC AND BIHARMONIC CANONICAL INTEGRAL EQUATIONS IN INTERIOR OR EXTERIOR CIRCULAR DOMAINS*

YU DE-HAO (余德浩)
(*Computing Center, Academia Sinica*)

Abstract

Elliptic boundary-value problems can be reduced to integral equations on the boundary by many different ways. The canonical reduction, suggested by Prof. Feng Kang[1-3], is a natural and direct approach of boundary reduction. This paper gives the numerical method for solving harmonic and biharmonic canonical integral equations in interior or exterior circular domains, together with their convergence and error estimates. Using the theory of distributions, the difficulty caused by the singularities of integral kernel is overcome. Results of several numerical calculations verify the theoretical estimates.

Introduction

In recent years, Feng Kang suggested a natural and direct method of redution, called canonical redution, of elliptic boundary-value problems over a domain to integral equations on the boundary, which preserves all the essential characteristics, including self-adjointness, coerciveness, variational functional, etc., of the original problem[1-3], The kernel of the resulting equation on the boundary, called the canonical integral equation, contains singularities of the type of the finite part of divergent integrals in the sense of the theory of distributions, which are of higher order than those of the usual Cauchy-type.

This paper gives the numerical method, based on the finite element approximations, for solving the canonical integral equations corresponding to the Neumann problems of harmonic and biharmonic equations over interior or exterior circular domains. Convergence and error estimates are also given. Several numerical experiments verify well the theoretical estimates and demonstrate the efficacy of the method.

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1. Harmonic Canonical Integral Equation

1.1. Numerical solution

Consider the harmonic equation in the interior or the exterior to the circle with radius R with Neumann condition

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$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = u_n(\theta) & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{P1}$$

where $u_n(\theta) \in H^{-\frac{1}{2}}(\Gamma)$ and satisfies the consistency condition

$$\int_\Gamma u_n(\theta)\, d\theta = 0.$$

It is equivalent to the variational problem

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ D(u,\ v) = \int_\Gamma u_n(\theta)\, v ds, \qquad \forall v \in H^1(\Omega), \\ D(u,\ v) = \iint\limits_\Omega \nabla u \cdot \nabla v\, dp \end{cases}$$

for the interior problem. For the exterior problem, the space $H^1(\Omega)$ is replaced by

$$W_0^1(\Omega) = \left\{ u \left| \frac{u}{\sqrt{x^2+y^2}\ln(1+x^2+y^2)},\ u_x,\ u_y \in L^2(\Omega) \right. \right\}.$$

Moreover, the problem (P1) can be reduced into the canonical integral equation, which contains a singular kernel, as follows[1]

$$u_n(\theta) = -\frac{1}{4\pi R} \int_0^{2\pi} \frac{1}{\sin^2 \frac{\theta-\theta'}{2}} u_0(\theta')\, d\theta' \equiv -\frac{1}{4\pi R \sin^2 \frac{\theta}{2}} * u_0(\theta),$$

where $*$ denots the convolution, which can be defined through Fourier series in the sense of generalized functions[7, 8]. Since

$$-\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{\sin^2 \frac{\theta}{2}}\, d\theta = 0,$$

the solution of above-mentioned integral equation is unique up to an additive constant. It also corresponds to the variational problem

$$\begin{cases} \text{Find } u_0(\theta) \in H^{\frac{1}{2}}(\Gamma) \text{ such that} \\ \bar{D}(u_0,\ v_0) \dot{=} \int_\Gamma u_n(\theta) v_0(\theta) ds, \qquad \forall v_0 \in H^{\frac{1}{2}}(\Gamma), \\ \bar{D}(u_0,\ v_0) = -\int_0^{2\pi}\int_0^{2\pi} \dfrac{1}{4\pi \sin^2 \frac{\theta-\theta'}{2}} u_0(\theta') v_0(\theta)\, d\theta'\, d\theta. \end{cases} \tag{1}$$

The original solution u of (P1) is obtained from u_0 by the Poisson formula

$$u(r,\ \theta) = \pm \frac{(R^2-r^2)}{2\pi} \int_0^{2\pi} \frac{u(R,\ \theta')}{R^2+r^2-2Rr\cos(\theta-\theta')}\, d\theta', \text{ for } \begin{cases} 0 \leqslant r < R, \\ r > R. \end{cases}$$

We can easily prove

Proposition 1.1. $\bar{D}(\gamma u,\ \gamma v) = D(u,\ v)$, $\forall u,\ v \in H^1(\Omega)$, $\Delta u = 0$; $\bar{D}(u_0,\ v_0)$ is a positive definite symmetric bilinear form on $[H^{\frac{1}{2}}(\Gamma)/P_0] \times [H^{\frac{1}{2}}(\Gamma)/P_0]$, where P_0