ORTHOGONAL PROJECTIONS AND THE PERTURBATION OF THE EIGENVALUES OF SINGULAR PENCILS*

SUN JI-GUANG (孙继广)

(Computing Center, Academia Sinica)

Abstract

In this paper we obtain a Hoffman–Wielandt type theorem and a Bauer–Fike type theorem for singular pencils of matrices. These results delineate the relations between the perturbation of the eigenvalues of a singular diagonalizable pencil $A-\lambda B$ and the variation of the orthogonal projection onto the column space $\mathfrak{R}\begin{pmatrix}A^H\\B^H\end{pmatrix}$.

1. Introduction

Let A and B be complex $m\times n$ matrices. A pencil of matrices $A-\lambda B$ is called singular if $m\neq n$ or $m=n$ but $\det(A-\lambda B)\equiv 0$[4]. A prevalent viewpoint is that in this case any complex number λ is an eigenvalue of $A-\lambda B$ (ref. [6]), consequently it is difficult to investigate the perturbation of the eigenvalues of singular pencils. In this paper we adopt a new definition for the eigenvalues of a singular pencil which is due to P. van Dooren[2], and relate the perturbation of the eigenvalues of $A-\lambda B$ and the variation of the orthogonal projection onto the column space $\mathfrak{R}\begin{pmatrix}A^H\\B^H\end{pmatrix}$ to each other, thus obtain a Hoffman–Wielandt type theorem (§ 3) and a Bauer–Fike type theorem (§ 4) for singular pencils which are generalizations of the main results for regular pencils in [8] and [3].

Notation: Capital case is used for matrices and lower case Greek letters for scalars. The symbol $\mathbb{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices. $\bar{A}$ and A^T stand for conjugate and transpose of A, respectively; $A^H=\bar{A}^T$. $I^{(n)}$ is the $n\times n$ identity matrix, and 0 is the null matrix. The matrix $|A|$ has elements $|a_{ij}|$ if $A=(a_{ij})$. $A>0$ ($\geqslant 0$) denotes that H is a positive definite (semi-positive definite) Hermitian matrix. The colume space of A is denoted by $\mathfrak{R}(A)$ and the null space by $N(A)$. $\mathfrak{R}(A)^\perp$ is the orthogonal complement space of $\mathfrak{R}(A)$. $G_{1,2}$ denotes the complex projective plane. The chordal distance between the points (α,β) and (γ,δ) on $G_{1,2}$ is

$$\rho((\alpha,\beta),(\gamma,\delta))=\frac{|\alpha\delta-\beta\gamma|}{\sqrt{(|\alpha|^2+|\beta|^2)(|\gamma|^2+|\delta|^2)}}.$$

* Received August 17, 1982.

1) This work was done while the author was visiting the University of Bielefeld (FRG) and assisted by the Alexander von Humboldt Foundation in the Federal Republic of Germany.

2) P. van Dooren has advanced a new definition for the eigenvalues of a singular pencil in his lecture "A numerical method to compute reducing subspaces of a singular pencil" at "The Conference on Matrix Pencils" in March 1982, Piteå, Sweden.

The matrix pencil formed of A and B can be expressed more precisely as $\mu A-\lambda B$, $(\lambda, \mu)\in G_{1,2}$.

2. Preliminaries

In this section we give some definitions and basic results.

2.1. Eigenvalues and eigenvectors

Definition 2.1. *Let $A, B\in\mathbb{C}^{m\times n}$, and* $\max_{(\lambda,\mu)\in G_{1,2}} \operatorname{rank}(\mu A-\lambda B)=k$. *A number-pair $(\alpha, \beta)\in G_{1,2}$ is an eigenvalue of the pencil $\mu A-\lambda B$ if* $\operatorname{rank}(\beta A-\alpha B)<k$.

The set of all eigenvalues of $\mu A-\lambda B$ is denoted by $\lambda(A, B)$.

The following consequences of Definition 2.1 can easily be verified.

i) If $\mu A-\lambda B$ is a regular pencil (i. e. $m=n$ and $\det(\mu A-\lambda B)\not\equiv 0$, $(\lambda, \mu)\in G_{1,2}$)[4], then Definition 2.1 is coincide with the usual definition[8].

ii) If $P\in\mathbb{C}^{m\times m}$ and $Q\in\mathbb{C}^{n\times n}$ are non-singular, then

$$\lambda(PAQ, PBQ)=\lambda(A, B). \tag{1.1}$$

Kronecker showed that if $\mu A-\lambda B$ is a singular pencil, then there exist non-singular matrices P and Q such that[4,10,2]

$$P(\mu A-\lambda B)Q=\begin{bmatrix} L_{\varepsilon_1}(\lambda,\mu) & & & & & \\ & \ddots & & & & \\ & & L_{\varepsilon_p}(\lambda,\mu) & & & \\ & & & L^T_{\eta_1}(\lambda,\mu) & & \\ & & & & \ddots & \\ & & & & & L^T_{\eta_q}(\lambda,\mu) & \\ & & & & & & \mu A_0-\lambda B_0\end{bmatrix}, \quad (\lambda,\mu)\in G_{1,2}. \tag{1.2}$$

Where $L_\varepsilon(\lambda, \mu)\in\mathbb{C}^{\varepsilon\times(\varepsilon+1)}$ and $L^T_\eta(\lambda, \mu)\in\mathbb{C}^{(\eta+1)\times\eta}$ are elementary Kronecker blocks, e. g. $L_2(\lambda, \mu)=\begin{pmatrix}\mu & -\lambda & 0\\ 0 & \mu & -\lambda\end{pmatrix}$; $\mu A_0-\lambda B_0$ is a regular pencil.

From the Kronecker's canonical form (1.2) it follows that

iii) $$\lambda(A, B)=\lambda(A_0, B_0).$$

We say that a singular pencil $\mu A-\lambda B$ contains an r-order regular part if $\mu A_0-\lambda B_0\in\mathbb{C}^{r\times r}$ in the form (1.2). The symbol $\mathfrak{S}^{m\times n}_r$ is used to denote the set of all $m\times n$ singular pencils, each of which contains an r-order regular part.

Definition 2.2. *Let $\mu A-\lambda B\in\mathfrak{S}^{m\times n}_r$. A non-zero vector $x\in\mathbb{C}^n$ is called an eigenvector of the singular pencil $\mu A-\lambda B$ corresponding to the eigenvalue (α, β) if*

$$\beta Ax=\alpha Bx, \quad (Ax, Bx)\neq(0, 0).$$

2.2. Singular diagonalizable pencils and singular normal pencils

Definition 2.3. *A pencil $\mu A-\lambda B\in\mathfrak{S}^{m\times n}_r$ is called diagonalizable if there exist r linearly independent eigenvectors $x_1, \cdots, x_r$ of $\mu A-\lambda B$ and a complement space $\mathfrak{R}(x_1, \cdots, x_r)^c$ of $\mathfrak{R}(x_1, \cdots, x_r)$ satisfying*

$$\mathfrak{R}(x_1, \cdots, x_r)^c\subseteq\mathcal{N}(A)\cap\mathcal{N}(B).$$

The set of all such pencils is denoted by $\mathscr{D}^{m\times n}_r$.

Definition 2.4. *A pencil $\mu A-\lambda B\in\mathfrak{S}^{m\times n}_r$ is called normal if there exist r ortho-*