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ERROR ESTIMATES FOR THE FINITE ELEMENT SOLUTIONS OF SOME VARIATIONAL INEQUALITIES WITH NONLINEAR MONOTONE OPERATOR*

WANG LIE-HENG[1] (王烈衡)
(*Computing Center, Academia Sinica*)

Abstract

In this paper, we estimate the error of the linear finite element solutions of the obstacle problem and the unilateral problem with monotone operator. We obtained $O(h)$ error bound for the obstacle problem and $O(h^{3/4})$ error bound for the unilateral problem. And if the solution u^* of the unilateral problem possesses more smoothness, then $O(h)$ error bound can be obtained in the same way as [2].

1. Introduction

In Brezzi, Hager and Raviart[2], the error estimates for the linear finite element solutions of the obstacle problem and the unilateral problem with linear V-elliptic operator have been obtained. Their results are the following: $O(h)$ error bounds for both the obstacle and unilateral problems with linear finite elements. Now in this paper, we obtained the same result for the obstacle problem with nonlinear monotone operator. For the unilateral problem with nonlinear monotone operator, we obtained $O(h^{3/4})$ error bound just as [5], and if the solution u^* of the unilateral problem possesses more smoothness, then $O(h)$ error bound holds in the same way as [2].

Let Ω denote a bounded convex open subset of $\mathbb{R}^2$, $\partial\Omega$ denote the boundary of Ω. Let $H^m(\Omega)$ be the usual Sobolev space [1] consisting of real value functions defined on Ω with derivatives through order m in $L^2(\Omega)$; the norm on $H^m(\Omega)$ is denoted by $\|\cdot\|_{m,\Omega}$. Let V be a Hilbert space with norm $\|\cdot\|$ and V' be the dual of V with norm $\|\cdot\|_*$, the pairing between V and V' be denoted by $\langle\cdot, \cdot\rangle$.

Let T be a (generally nonlinear) mapping

$$T: V \mapsto V',$$

which possesses the two following properties (c. f. [3]):

(i) The mapping T is uniformly monotone, i. e., there exists a positive constant $\alpha>0$, such that

$$\langle Tu-Tv, u-v\rangle \geqslant \alpha\|u-v\|^2, \ \forall u, v \in V. \tag{1.1}$$

(ii) The mapping T is Lipschitz-continuous for bounded arguments in the sense

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that, for any ball $B(0; r)=\{v\in V: \|v\|\leqslant r\}$, there exists a constant $\Gamma(r)$ such that

$$\|Tu-Tv\|_*\leqslant\Gamma(r)\cdot\|u-v\|,\ \forall u,\ v\in B(0;r). \tag{1.2}$$

We state a well known result of interpolation [3, p. 124]: Let $v\in H^{k+1}(\Omega^h)$, v^I be the piecewise linear interpolation of v on Ω^h, and Ω^h be a regular triangulation[3], then

$$\|v-v^I\|_{m,\Omega^h}\leqslant Ch^{k+1-m}|v|_{k+1,\Omega},\ \text{for } k=1,\ m=0,\ 1, \tag{1.3}$$

$$\|v-v^I\|_{1,\Omega^h}\leqslant C|v|_{1,\Omega},\ \forall v\in H^1(\Omega), \tag{1.3'}$$

where C is a constant independent of h and v.

2. The Obstacle Problem

Let us introduce some other notations:

$$a(u,\ v)=\int_\Omega\left[a_1(u,\ \nabla u)\frac{\partial v}{\partial x_1}+a_2(u,\ \nabla u)\frac{\partial v}{\partial x_2}+a_0(u,\ \nabla u)v\right]dx_1\,dx_2, \tag{2.1}$$

$$\langle f,\ v\rangle=\int_\Omega f\cdot v\,dx_1dx_2, \tag{2.2}$$

$$K=\{v\in H^1(\Omega): v\geqslant\psi \text{ a. e. in } \Omega,\ v|_{\partial\Omega}=g\}. \tag{2.3}$$

Let us assume that $a_i(\xi_0,\ \xi_1,\ \xi_2)\in H^1(\mathbb{R}^3)$, $(i=0,\ 1,\ 2)$ and $a_{ij}=\frac{\partial}{\partial\xi_j}a_i(\xi)$, and

$$\sum_{i,j=0}^{2}a_{ij}(\xi)\eta_i\eta_j\geqslant\alpha\|\eta\|^2,\ \|\eta\|^2=\sum_{i=0}^{2}\eta_i^2,\ \forall\xi,\ \eta\in\mathbb{R}^3, \tag{2.4}$$

$$|\eta^T[a_{ij}(v,\ \nabla v)]_{ij}\eta|\leqslant\Gamma(r)\|\eta\|^2,\ \forall v\in H^1(\Omega);\ \|v\|_{1,\Omega}\leqslant r,\ \eta\in\mathbb{R}^3. \tag{2.5}$$

Then there exists an operator T,

$$T: H^1(\Omega)\mapsto(H^1(\Omega))',$$

defined by

$$a(u,\ v)=\langle Tu,\ v\rangle. \tag{2.6}$$

We can find that the mapping T defined above possesses two properties (i) and (ii) in section 1.

Let L denote an operator defined by

$$Lu=-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}a_i(u,\ \nabla u)+a_0(u,\ \nabla u). \tag{2.7}$$

The obstacle problem is to find $u^*\in K$, such that

$$a(u^*,\ v-u^*)\geqslant\langle f,\ v-u^*\rangle,\quad \forall v\in K,$$

or we can write it in the another form: to find $u^*\in K$, such that

$$\langle Tu^*,\ v-u^*\rangle\geqslant\langle f,\ v-u^*\rangle,\quad \forall v\in K. \tag{2.8}$$

If $f\in L^2(\Omega)$, $\psi\in H^2(\Omega)$, g is the restriction to $\partial\Omega$ of an $H^2(\Omega)$ function and $g\geqslant\psi$ on $\partial\Omega$, then the existence and unique of the solution of the problem (2.8) are insured by classical result[5].

If the solution $u^*\in H^2(\Omega)$, $\psi\in H^2(\Omega)$, $f\in L^2(\Omega)$ and $Lu^*\in L^2(\Omega)$, then the following differential forms holds[7]:

$$\begin{cases} Lu^*-f\geqslant0,\ (Lu^*-f)(u^*-\psi)=0,\ u^*\geqslant\psi \text{ a. e. in } \Omega, \\ u^*|_{\partial\Omega}=g. \end{cases} \tag{2.9}$$