ESTIMATION FOR SOLUTIONS OF ILL-POSED CAUCHY PROBLEMS OF DIFFERENTIAL EQUATIONS WITH PSEUDO-DIFFERENTIAL OPERATORS*

Part I. Case of First Order Operators

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Abstract

In this paper we discuss the estimation for solutions of the ill-posed Cauchy problems of the following differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\ \forall t\in(0,1),$$

where $A(t)$ is a p. d. o. (pseudo-differential operator (*s*)) of order 1 or 2, $N(t)$ is a uniformly bounded $H\to H$ linear operator. It is proved that if the symbol of the principal part of $A(t)$ satisfies certain algebraic conditions, two estimates for the solution $u(t)$ hold. One is similar to the estimate for analytic functions in the Three-circle Theorem of Hadamard. Another is the estimate of the growth rate of $\|u(t)\|$ when $A(1)u(1)\in H$.

Introduction

In this paper we will discuss the estimation for solutions of the Cauchy problems of the differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\ \forall t\in(0,1), \tag{1}$$

with the prescribed $u(0)=u_0$, where $u(t)=u(t,x)$ is a n-dimensional vector function, $x\in R^m$, $A(t)$ is a p. d. o. dependent on the parameter t, $N(t)$ is a uniformly (respectively to t) bounded linear operator $H\to H$. This Cauchy problem in general is not well-posed.

The simplest examples are the Cauchy problem of the Cauchy-Riemann equations and that of the backward heat equation. The estimate of solutions of the Cauchy problem of the Laplace equation was obtained by M. M. Lavrentiev[1]. The same estimate for the Cauchy-Riemann equations, the backward heat equation and that of (1), in which A is a differential operator with constant coefficients satisfying certain conditions were obtained in [2, 3, 4]. This estimate can be represented in the form

$$\|u(t)\|\leqslant c\|u(0)\|^{1-t}\|u(1)\|^t, \tag{2}$$

where c is a constant independent of $u(t)$.

The estimate (2) is significant in the investigation of approximate methods for solving the ill-posed Cauchy problems. In [4] the author discussed the difference

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schemes for solving (1), in which the differential operator A has constant coefficients. Considering the estimate (2), we gave an appropriate definition of stability, and proved the theorem of equivalence between convergence and stability for the consistent and commutative (with A) difference schemes. In [10] the finite element method for solving the Cauchy problems of the Laplace equation was discussed. So in order to investigate the approximate methods for (1) in the general case, it is important to establish the estimate of the solutions, similar to (2).

When $m=1$ and A is a first order differential operator whose coefficient matrix is variable and of simple structure, similar estimate was obtained by the author in terms of the Three-line Theorem[5]. It is very like the Three-circle Theorem of Hadamard. This estimate can be written in the form

$$\|u(t)\| \leqslant C \max_{i=1,2} \{\|u(0)\|^{1-\delta_i(t)} \|u(t)\|^{\delta_i(t)}\}, \tag{3}$$

where $\delta_i(t)$ are increasing functions of t, satisfying $\delta_i(0)=0$ and $\delta_i(1)=1$. If $(u(1), u(1))' = \frac{d}{dt}(u(t), u(t))|_{t=1}$ exists, then the growth rate of $\|u(t)\|$ can be estimated by

$$\|u(t)\| \leqslant c^* \|u(0)\| e^{K(u(1))t}, \tag{4}$$

where $K(u(1)) = c^{**}(u(1), u(1))'/(u(1), u(1))$. c, c^* and c^{**} are all constant, independent of $u(t)$.

In [5], the author discussed the second order ordinary differential inequalities and obtained two estimates for the solutions. One of them is an extension of the inequality for convex functions. These two inequalities play an important role in the estimation of the solutions of ill-posed Cauchy problems. We shall reformulate the lemma about these two inequalities (in section 2) and use them to prove the main theorems of this paper. Using the same inequalities, S. Agmon and L. Nirenberg[6] obtain the estimation for solutions of abstract differential equations in a Hilbert space. In [7] estimation of this type was also discussed. Many results about estimation for solutions of ill-posed problems in P. D. E. are outlined in [8].

The aim of this paper is to estimate the solutions of (1), in which $A(t)$ is a first order or second order p. d. o.. We shall prove that estimates similar to (3) and (4) are also valid for the solutions of (1), when the symbol of the principal part of $A(t)$ satisfies certain easily verified algebraic conditions. As mentioned above, the estimate (3) is meaningful to approximate methods, this paper can be considered as a preparation for the discussion of approximate methods of ill-posed problem (1).

This paper is divided into two parts. Part 1 is devoted to the case of first order p. d. o.. First, we review briefly some theorems about p. d. o.. Then we formulate some lemmas, which are used in the proof of the main theorems of this paper. One of them is on the second order ordinary differential inequalities. The others describe truncators and quasi-inverses. Finally we derive the estimates for solutions of (1). In part II we discuss the case, in which $A(t)$ is of second order.

§ 1. Pseudo-Differential Operators

In this section we first recall the definition of p. d. o. on vector-valued functions $u(x)$ and some theorems about them, according to [9].