COUPLING CANONICAL BOUNDARY ELEMENT METHOD WITH FEM TO SOLVE HARMONIC PROBLEM OVER CRACKED DOMAIN*

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Abstract

Using the canonical boundary reduction, suggested by Feng Kang [1,2], coupled with the finite element method, this paper gives the numerical solution of the harmonic boundary-value problem over the domain with crack or concave angle. When the coupling is conforming, convergence and error estimates are obtained. This coupling removes the limitation of the canonical boundary reduction to some typical domains, and avoids the shortcoming of the classical finite element method, because of which the accuracy is damaged seriously and the approximate solution does not reflect the behaviour of the solution near the singularity. Numerical calculations have verified those conclusions.

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It is known that elliptic boundary-value problems can be reduced to integral equations on the boundary in different ways. The canonical reduction, suggested by Feng Kang in recent years [1,2], is a natural and direct reduction, which preserves the essential characteristics of the original problem. Unfortunately, it is only applicable to some typical domains. The classical finite element method can be applied to relatively arbitrary domains, but except cracked domains. Therefore it is only natural to couple the canonical boundary element method with the finite element method.

1. The Method

In [3], a numerical method by canonical boundary reduction with error estimates is given for solving two kinds of boundary-value problems of harmonic equation over sector with crack and concave angle. For the harmonic boundary-value problem over general domain with crack and concave angle, we can couple the canonical boundary element method with the finite element method as follows.

Let Ω be a domain bounded by two sides Γ_1 and Γ_2 of a concave angle α ($\pi<\alpha\leqslant 2\pi$) and a smooth curve Γ. When $\alpha=2\pi$, the domain contains a crack. Consider the boundary-value problem

$$\begin{cases} \Delta u=0, \text{ in } \Omega, \\ \partial_n u=0, \text{ on } \Gamma_1\cup\Gamma_2, \qquad \partial_n u=f, \text{ on } \Gamma, \end{cases} \tag{1}$$

where $f\in H^{-\frac{1}{2}}(\Gamma)$ satisfies the consistency condition

$$\int_\Gamma f ds=0.$$

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It is well-known, that problem (1) has a solution unique up to a constant. Let

$$D(u, v)=\iint_{\Omega}\nabla u\cdot\nabla v\,dp.$$

Then problem (1) is equivalent to the variational problem

$$\begin{cases}\text{Find } u\in H^1(\Omega)/P_0 & \text{such that}\\ D(u, v)=\int_{\Gamma}vf\,ds, & \forall v\in H^1(\Omega),\end{cases}\tag{2}$$

where P_0 is all constants. Using the Lax-Milgram lemma, we can easily prove that problem (2) has one and only one solution in $H^1(\Omega)/P_0$.

Now take the vertex of angle α as origin and place Γ_1 on x axis. Then draw in Ω an arc $\Gamma'=\{(R, \theta)\,|_{0<\theta<\alpha}\}$ dividing Ω into Ω_1 and Ω_2, where Ω_2 is a sector. We have

$$\iint_{\Omega}\nabla u\cdot\nabla v\,dp=\iint_{\Omega_1}\nabla u\cdot\nabla v\,dp+\int_{\Gamma'}v\partial_n u\,ds,$$

and

$$u_n(\theta)=-\frac{\pi}{4\alpha^2R}\int_0^{\alpha}\left(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi}+\frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\right)u(R, \theta')d\theta', \quad 0<\theta<\alpha,$$

which is the canonical integral equation of Γ' obtained in [3]. It contains a singular kernel and can be defined in the sense of distributions. Then problem (1) is equivalent to the variational problem

$$\begin{cases}\text{Find } u\in H^1(\Omega_1)/P_0 \quad \text{such that}\\ D_1(u, v)+\overline{D}_2(\gamma' u, \gamma' v)=\int_{\Gamma}vf\,ds, \quad \forall v\in H^1(\Omega_1),\end{cases}\tag{3}$$

where $D_1(u, v)=\iint_{\Omega_1}\nabla u\cdot\nabla v\,dp$,

$$\overline{D}_2(u_0, v_0)=-\frac{\pi}{4\alpha^2}\int_0^{\alpha}\left(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi}+\frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\right)u_0(\theta')v_0(\theta)d\theta'\,d\theta,$$

γ' is the trace operator maping $H^1(\Omega_1)$ onto $H^{\frac{1}{2}}(\Gamma')$.

From the existence and uniqueness of the solution of the variational problem (2) the following is immediate.

Proposition 1. The variational problem (3) has one and only one solution in $H^1(\Omega_1)/P_0$.

Now divide arc Γ' into N_1 and subdivide Ω_1 into triangles such that its nodes on Γ' coincide with the dividing points of Γ'. Let $\{L_i(x, y)\}_{i=0}^{N_1+N_2}\subset H^1(\Omega_1)$ be basis functions, for example, piecewise linear; then their restrictions on Γ' are approximately piecewise linear on Γ'. Let

$$u\approx U(x, y)=\sum_{i=0}^{N_1+N_2}U_iL_i(x, y),$$

where the subscripts $i=0, 1, \cdots, N_1$ correspond to the nodes on Γ'. We have

$$\sum_{j=0}^{N_1+N_2}D_1(L_j, L_i)U_j+\sum_{j=0}^{N_1}\overline{D}_2(\gamma' L_j, \gamma' L_i)U_j=\int_{\Gamma}fL_i\,ds, \quad i=0, 1, \cdots, N_1+N_2,$$

or, for simplicity,

$$QU=b.\tag{4}$$