THE MONOTONICITY PROBLEM IN FINDING ROOTS OF POLYNOMIALS BY KUHN'S ALGORITHM*

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Abstract

In this paper the problem proposed by Kuhn on the presence of a monotonicity property related to the Kuhn's algorithm for finding roots of a polynomial is solved in the affirmative. Furthermore, an estimate of the threshold number D in the above-mentioned monotonicity problem expressed in terms of the complex coefficients of the polynomial is obtained.

Introduction

Kuhn has constructed in [1] the sequences (z_{jk}, d_{jk}), $j=1, \cdots, n$, $k=1, 2, \cdots$; $\lim_{k\to+\infty} z_{jk}=\tilde{z}_j$; $\tilde{z}_1, \cdots, \tilde{z}_n$ are the roots of a monic polynomial $f(z)$ of degree n in the complex variable z with complex numbers as coefficients. He recently posed a monotonicity problem: If $\tilde{z}_1, \cdots, \tilde{z}_n$ are the simple roots of $f(z)$, does there exist a number D such that when $d_{jk}\geqslant D$, both d and $d_{jk}+d_{jk'}+d_{jk''}+d_{jk'''}$ are increasing; (z_{jk}, d_{jk}) belongs to a tetrahedron $\{(z_{jk}, d_{jk}), (z_{jk'}, d_{jk'}), (z_{jk''}, d_{jk''}), (z_{jk'''}, d_{jk'''})\}$, $k>k', k'', k'''$ and $d\leqslant d_{jk}, d_{jk'}, d_{jk''}, d_{jk'''}\leqslant d+1$.

He further asked how to find the expression of D in a_j, $a_1, \cdots, a_n$ being the complex coefficients of $f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$, and how to find D such that when $d\geqslant D$, there is just one triangle labelled (1, 2, 3) in $U((\tilde{z}_j, L)\subset C_d$, where $U(\tilde{z}_j, L)$, $j=1, \cdots, n$, are disjoint open circular discs.

This paper aims to answer these problems.

1. A Monotonicity Problem

Lemma 1.1. *If* $|z|>\max_k|a_k|+1$, *then* $f(z)\neq 0$, *that is,* $\max_k|\tilde{z}_k|\leqslant\max_k|a_k|+1$, *where* $\tilde{z}_1, \cdots, \tilde{z}_n$ *are the roots of* $f(z)$.

Proof. Since

$$|f(z)|=\left|z^n\left(1+\sum_{l=1}^n \frac{a_l}{z^l}\right)\right|\geqslant|z^n|\left(1-\sum_{l=1}^n \frac{|a_l|}{|z|^l}\right)$$

$$\geqslant|z^n|\left(1-\max_k|a_k|\sum_{l=1}^\infty \frac{1}{|z|^l}\right)=|z^n|\left(1-\frac{\max_k|a_k|}{|z|-1}\right)>0,$$

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therefore
$$f(z)\neq 0.$$

Let
$$\varphi(z)=\sum_{l=0}^{n}|a_l|z^l,$$
$$R=\max_k|a_k|+1,$$
$$M=1+\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{(l-1)!}.$$

Then
$$|f^{(s)}(\tilde z_j)|=\left|\sum_{l=s}^{n}l(l-1)\cdots(l-s+1)a_l\tilde z_j^{l-s}\right|$$
$$\leqslant\sum_{l=s}^{n}l(l-1)\cdots(l-s+1)|a_l|R^{l-s}=\varphi^{(s)}(R),$$
$s=1,\cdots,n.$

Lemma 1.2. *Let $\tilde z_1,\cdots,\tilde z_n$ be the simple roots of $f(z)$; $0<N\leqslant\min_k|f'(\tilde z_k)|$; $\{z_1, z_2, z_3\}$ is a triangle in C_{d+1} of a special triangulation in* [1] (see Figure 4). *If $\max_k|z_k-\tilde z_j|\leqslant\min\left\{1,\ \frac{N}{5M}\right\}=\sigma$ for some $\tilde z_j$, then $\{z_1, z_2, z_3\}$ is not labelled* (1, 3, 2).

Proof. Since $f(\tilde z_j)=0$, according to Taylor's formula,
$$f(z)=f'(\tilde z_j)(z-\tilde z_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}(z-\tilde z_j)^l,$$
and we obtain
$$\frac{f(z_2)-f(z_3)}{f(z_1)-f(z_3)}=\frac{f'(\tilde z_j)(z_2-z_3)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}[(z_2-\tilde z_j)^l-(z_3-\tilde z_j)^l]}{f'(\tilde z_j)(z_1-z_3)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}[(z_1-\tilde z_j)^l-(z_3-\tilde z_j)^l]}$$
$$=\frac{z_2-z_3}{z_1-z_3}\left[1+\frac{\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}\sum_{s=1}^{l}(z_2-\tilde z_j)^{l-s}(z_3-\tilde z_j)^{s-1}-\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde z_j)^{l-s}(z_3-\tilde z_j)^{s-1}}{f'(\tilde z_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde z_j)^{l-s}(z_3-\tilde z_j)^{s-1}}\right].$$

When $\max_k|z_k-\tilde z_j|\leqslant\min\left\{1,\ \frac{N}{5M}\right\}=\sigma$, we have
$$\left|\frac{\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l}\sum_{s=1}^{l}(z_2-\tilde z_j)^{l-s}(z_3-\tilde z_j)^{s-1}-\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde z_j)^{l-s}(z_3-\tilde z_j)^{s-1}}{f'(\tilde z_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde z_j)^{l-s}(z_3-\tilde z_j)^{s-1}}\right|$$
$$\leqslant\frac{2\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{l!}l\sigma^{l-1}}{|f'(\tilde z_j)|-\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{l!}l\sigma^{l-1}}\leqslant\frac{2\sigma\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{(l-1)!}}{N-\sigma\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{(l-1)!}}=\frac{2\sigma M}{N-\sigma M}\leqslant\frac{1}{2},$$
so
$$\frac{\pi}{12}=\frac{\pi}{4}-\frac{\pi}{6}\leqslant\arg\frac{z_2-z_3}{z_1-z_3}-\frac{\pi}{6}\leqslant\arg\frac{f(z_2)-f(z_3)}{f(z_1)-f(z_3)}\leqslant\arg\frac{z_2-z_3}{z_1-z_3}+\frac{\pi}{6}\leqslant\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$$
(see Figure 1).

If $\{z_1, z_2, z_3\}$ is a triangle labelled (1, 3, 2), without loss of generality, we only have to show the case of Figure 2. Then it follows that