ON THE CONVERGENCE RATE OF THE BOUNDARY PENALTY METHOD* 1)

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Abstract

The convergence rate of the boundary penalty finite element method is discussed for a model Poisson equation with inhomogeneous Dirichlet boundary conditions and a sufficiently smooth solution. It is proved that an optimal convergence rate can be achieved which agrees with the rate obtained recently in the numerical experiments by Utku and Carey.

1. Introduction

The boundary penalty finite element method has been developed to approximate Dirichlet boundary conditions in the solution of elliptic boundary value problems. (see, for example, references [2] to [5].) In finite element programs there exists another technique for approximating Dirichlet data by adding a large number to certain diagonal entries in the stiffness matrix and by scaling the load vector. Utku and Carey [1] have recently discussed the relationship between these two techniques for treating Dirichlet data and derived an abstract error estimate for the boundary penalty method. They observed that the rates achieved in the numerical experiments are better than those obtained in the theoretical analysis. In this note we show, using the regularity of the solution of the given problem, that the theoretical result of Utku and Carey can be improved, and we present an error estimate which fully agrees with the numerical experiments in [1].

2. A Model Problem

As a model problem, consider the solution of Poisson's equation

$$-\Delta u=f \quad \text{in } \Omega \tag{1}$$

with inhomogeneous Dirichlet data

$$u=g \quad \text{on } \partial\Omega. \tag{2}$$

Suppose that $g\in H^{\frac{1}{2}}(\partial\Omega)$. Then, by the trace theorem (see [6, p. 143]), there exists a function $\tilde{g}\in H^1(\Omega)$ such that $|\tilde{g}|_{\partial\Omega}=g$. The inhomogeneous Dirichlet problem (1), (2) is equivalent to the following variational equation: find $u\in H^1(\Omega)$ such that

$$u-\tilde{g}\in H_0^1(\Omega), \quad a(u, v)=(f, v), \quad v\in H_0^1(\Omega), \tag{3}$$

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where

$$a(u, v)=\int_{\Omega} \nabla u \cdot \nabla v \, dx\, dy, \qquad (f, v)=\int_{\Omega} f v \, dx\, dy. \tag{4}$$

Assuming the given domain Ω to be a convex polygon, we decompose it into triangular elements K satisfying the usual regularity conditions for triangulations. For simplicity, consider the continuous piecewise linear function spaces $V_h \in H^1(\Omega)$ and let Π_1 be the interpolation operator of functions $u \in H^2(\Omega)$ at the vertices of the triangles K in $\bar{\Omega}$, such that

$$\Pi_1 u \in V_h, \quad u \in H^2(\Omega). \tag{5}$$

From interpolation theory the following estimates are obtained:

$$\|u-\Pi_1 u\|_{0,K} \leqslant Ch^2 |u|_{2,K}, \tag{6}$$

$$|u-\Pi_1 u|_{1,K} \leqslant Ch |u|_{2,K}, \tag{7}$$

for $u \in H^2(\Omega)$, and

$$\int_{\partial K} w^2 \, ds \leqslant C_1 h |w|_{1,K}^2 + \frac{C_2}{h} \|w\|_{0,K}^2 \tag{8}$$

for $w \in H^1(\Omega)$, uniformly for all elements K. Here and in the following all C_i and C denote generic constants independent of h. As direct consequences of the inequalities (6), (7), (8), we have for $u \in H^2(\Omega)$:

$$\|u-\Pi_1 u\|_1^2 \leqslant Ch^2 |u|_2^2, \tag{9}$$

$$\int_{\partial\Omega} (u-\Pi_1 u)^2 \, ds \leqslant Ch^3 |u|_2^2. \tag{10}$$

The penalty method for approximating the solution of the variational equation (3) by the finite element spaces V_h consists in finding $u_h \in V_h$ such that

$$a(u_h, v_h) + h^{-\sigma} \int_{\partial\Omega} (u_h - g) v_h \, ds = (f, v_h), \quad v_h \in V_h, \tag{11}$$

where $\sigma > 0$ is a penalty parameter to be determined in order to maximize the rate of convergence.

3. H^2-Solution

Lemma 1. *Let u_0, u_h be the solutions of equations* (3), (11), *respectively, and suppose that $u_0 \in H^2(\Omega)$. Then the inequality*

$$|u_0-u_h|_1^2 + h^{-\sigma} \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n} h^{\sigma} + u_h - g\right)^2 ds \leqslant |u_0-v_h|_1^2 + h^{-\sigma} \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n} h^{\sigma} + v_h - g\right)^2 ds \tag{12}$$

holds for all $v_h \in V_h$.

Proof. Since u_h is the solution of the variational equation (11),

$$a(u_h, u_h) + h^{-\sigma} \int_{\partial\Omega} (u_h - g)^2 \, ds - 2(f, u_h) \leqslant a(v_h, v_h) + h^{-\sigma} \int_{\partial\Omega} (v_h - g)^2 \, ds - 2(f, v_h) \tag{13}$$

for $v_h \in V_h$. On the other hand, it follows from (1), (2) that