# NONLINEAR IMPLICIT ONE-STEP SCHEMES FOR SOLVING INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATION WITH STEEP GRADIENTS*

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**Abstract**

A general theory for nonlinear implicit one-step schemes for solving initial value problems for ordinary differential equations is presented in this paper. The general expansion of "symmetric" implicit one-step schemes having second-order is derived and stability and convergence are studied. As examples, some geometric schemes are given.

Based on previous work of the first author on a generalization of means, a fourth-order nonlinear implicit one-step scheme (GMS) is presented for solving equations with steep gradients. Also, a hybrid method based on the GMS and a fourth-order linear scheme is discussed. Some numerical results are given.

## 1. Introduction

Many classical methods for solving initial value problems for ordinary differential equations are based on piecewise polynomial interpolation. If the solution of the problem possesses a very steep gradient, these schemes produce poor results. In particular, if a singularity occurs, it is often inappropriate to attempt to represent the solution in the neighborhood of the singularity by a polynomial. In this paper, we consider a class of nonlinear implicit one-step schemes that may be more appropriate for such problems.

A general theory for nonlinear implicit one-step schemes is developed in Section 2. Conditions for consistency, stability and convergence are obtained. Every consistent symmetric scheme is at least second-order, and the condition for them to be fourth-order is given. A class of symmetric and homogeneous schemes which are generalizations of the well-known trapezoidal rule is obtained.

The trapezoidal rule is exact for second-degree polynomials. In terms of geometry, a second-degree polynomial is a conic. As examples of nonlinear symmetric implicit schemes, we develop several geometric schemes based upon "circles", "ellipses", "parabolae", and "hyperbolae" in Section 3.

On the other hand, in terms of means, the trapezoidal rule is the arithmetic mean of the first derivative of the solution at two neighboring grid points. In Section 4, based on the generalization of means[9], a fourth-order nonlinear implicit one-step scheme (GMS) which is shown to be efficient in numerical tests is presented for solving problems with steep gradients.

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In Section 5, we discuss some practical considerations including the use of hybrid methods based upon the GMS and more traditional schemes.

In this paper, the theory of nonlinear implicit one-step schemes is restricted to scalar equations. However, we have used these schemes successfully to solve systems of equations. The application of these schemes to systems is discussed briefly in Section 6.

Numerical results for seven test problems, some of which contain systems of equations, are given in the last section. Two of the examples use an imbedding technique to apply the GMS to the solution of two-point boundary value problems.

## 2. A General Theory for Nonlinear Implicit One-step Schemes

Consider the initial value problem (I. V. P.)

$$y'=f(x, y), \quad y(a)=y_0 \quad (a\leqslant x\leqslant b), \tag{1}$$

where $f(x, y)$ is continuous in $x$ and Lipschitz continuous in $y$ in the region $a\leqslant x\leqslant b$, $-\infty<y<\infty$, $a$ and $b$ finite.

We investigate the following general nonlinear implicit one-step scheme

$$Y_{n+1}=Y_n+hS(f_n, f_{n+1}), \tag{2}$$

where $\quad h=x_{n+1}-x_n, \quad f_n=f(x_n, Y_n), \quad f_{n+1}=f(x_{n+1}, Y_{n+1}).$

The local truncation error for the scheme (2) is

$$L(f)\equiv y(x_{n+1})-y(x_n)-hS(f(x_n, y(x_n)), f(x_{n+1}, y(x_{n+1}))), \tag{3}$$

where $y(x)$ is the solution of (1).

**Definition 1**[5]. *The scheme* (2) *is said to have order $p$ if $p$ is the largest integer for which*

$$L(f)=O(h^{p+1}).$$

**Definition 2.** *The scheme* (2) *is said to be consistent with the I. V. P.* (1) *if* $L(f)=o(h)$.

We will use the notation $f(t)\equiv f(t, y(t))$ throughout this paper except where it may be confused. For $x_n\leqslant x\leqslant x_{n+1}$, let $t=(x-x_n)/h$. Since

$$y(x_{n+1})-y(x_n)=h\int_0^1 f(x_n+th)dt,$$

(3) may be rewritten as

$$L(f)=h\left\{\int_0^1 f(x_n+th)dt-S(f(x_n, y(x_n)), f(x_{n+1}, y(x_{n+1})))\right\}. \tag{4}$$

By the Integral Mean Value Theorem, there exists a point $\xi$ between $x_n$ and $x_{n+1}$ such that

$$\int_0^1 f(x_n+th)dt=f(\xi, y(\xi)).$$

So $$\frac{L(f)}{h}=f(\xi, y(\xi))-S(f(x_n, y(x_n)), f(x_{n+1}, y(x_{n+1}))).$$

Furthermore, if $f'(t)=\frac{df}{dt}$ exists, then

$$\int_0^1 f(x_n+th)dt=f(x_n)+\int_0^1 f'(x_n+th)(1-t)dt$$