GENERALIZED BERNSTEIN-BÉZIER POLYNOMIALS*

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In Computer Aided Geometric Design, the following functions

$$f_{n,0}(x)=1,$$
$$f_{n,i}(x)=\frac{(-x)^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left[\frac{(1-x)^n-1}{x}\right],\quad i=1,\ 2,\ \cdots,\ n \tag{1}$$

are known as the nth Bézier basis functions [1],[2]. The analytical properties of these functions have been studied by many authors. It is proved in [3] that

$$f_{n,i}(x)=J_{n,i}(x)+J_{n,i+1}(x)+\cdots+J_{n,n}(x), \tag{2}$$

where $J_{n,i}$ stands for $\binom{n}{i}x^i(1-x)^{n-i}$, $i=0,\ 1,\ \cdots,\ n$, the nth Bernstein basis function. Simple calculations show that

$$f_{n,i}(x)-f_{n,i+1}(x)=J_{n,i}(x), \tag{3}$$
$$f'_{n,i}(x)=nJ_{n-1,i-1}(x),\quad i=1,\ 2,\ \cdots,\ n. \tag{4}$$

It is clear from (3) and (4) that

$$f_{n,1}(x)>f_{n,2}(x)>\cdots>f_{n,n}(x),\quad x\in(0,\ 1), \tag{5}$$

and that $f_{n,i}(x)$, $i=1,\ 2,\ \cdots,\ n$, increases strictly from 0 to 1 on [0, 1].

For each function $\varphi(x)$ defined on [0, 1] and each real number $\alpha>0$, we define

$$B_{n,\alpha}(\varphi;x)=\varphi(0)+\sum_{i=1}^{n}\left[\varphi\left(\frac{i}{n}\right)-\varphi\left(\frac{i-1}{n}\right)\right]f^{\alpha}_{n,i}(x), \tag{6}$$

or equivalently

$$B_{n,\alpha}(\varphi;x)=\sum_{i=0}^{n}\varphi\left(\frac{i}{n}\right)[f^{\alpha}_{n,i}(x)-f^{\alpha}_{n,i+1}(x)], \tag{7}$$

where $f_{n,n+1}=0$. In the case $\alpha=1$, we see from (7) and (3) that $B_{n,1}(\varphi;x)$ is just the nth Bernstein polynomial of $\varphi(x)$. (6) and (7) are called the generalized Bernstein-Bézier polynomial of $\varphi(x)$, although they may fail to be polynomials when α is not a positive integer.

In this paper, the uniform convergence

$$\lim_{n\to\infty}B_{n,\alpha}(\varphi;x)=\varphi(x)$$

is established for $\varphi(x)$ continuous on [0, 1] and for each $\alpha>0$. And a theorem similar to that of Kelisky and Rivlin for the iterates of Bernstein operators is proved.

A proof of the uniform convergence of $B_{n,\alpha}(\varphi)$ is also given, which is elementary but rather tedious. Professor Chen Xiru points out that $f_{n,i}(x)$ represents the probability that an event A occurs i or more than i times in n independent trials, where x is the probability that A occurs in a given trial, as shown by (2). He also

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indicates that by the Tchebichev inequality ([6], p. 11) we have for arbitrarily given $\varepsilon>0$

$$\lim_{n\to\infty} f_{n,i}(x)=\begin{cases}1, & \text{for } i\leqslant n(x-\varepsilon),\\ 0, & \text{for } i\geqslant n(x+\varepsilon)\end{cases} \tag{8}$$

uniformly for $x\in[0, 1]$, and that the first lemma of this paper follows immediately from the fact that $0\leqslant f_{n,i}(x)\leqslant 1$.

The following two identities are useful in the sequel:

$$\frac{1}{n}\sum_{i=1}^{n} f_{n,i}(x)=x, \tag{9}$$

$$\frac{1}{n^2}\sum_{i=1}^{n} i f_{n,i}(x)=\frac{x}{n}+\left(1-\frac{1}{n}\right)\frac{x^2}{2}. \tag{10}$$

Since by (4) we have

$$\sum_{i=1}^{n} f'_{n,i}(x)=n\sum_{i=1}^{n} J_{n-1,i-1}(x)=n$$

and $f_{n,i}(0)=0$, (9) is proved. Similarly we have

$$\sum_{i=1}^{n} i f'_{n,i}(x)=n\sum_{i=1}^{n} iJ_{n-1,i-1}(x)=n\sum_{i=1}^{n} i[f_{n-1,i-1}(x)-f_{n-1,i}(x)]$$

$$=n\sum_{i=0}^{n-1} f_{n-1,i}(x)=n[1+(n-1)x].$$

Hence (10) follows. We are going to prove the following

Lemma 1. *For each real number $\alpha>0$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f^{\alpha}_{n,i}(x)=x \tag{11}$$

uniformly in $[0, 1]$.

Proof. Assume $\alpha\geqslant 1$. For arbitrarily given real numbers $\varepsilon>0$ and $\delta>0$, there exists a positive integer $N=N(\varepsilon, \delta)$ by (8) such that

$$0\leqslant 1-f^{\alpha-1}_{n,i}(x)<\delta, \quad \text{if } i\leqslant n(x-\varepsilon),$$

$$0\leqslant f_{n,i}(x)<\delta, \quad \text{if } i\geqslant n(x+\varepsilon),$$

for $x\in[0, 1]$ and $n>N$. Hence we have

$$0\leqslant x-\frac{1}{n}\sum_{i=1}^{n} f^{\alpha}_{n,i}(x)=\frac{1}{n}\sum_{i=1}^{n} f_{n,i}(x)[1-f^{\alpha-1}_{n,i}(x)]$$

$$=\frac{1}{n}\Big[\sum_{i<n(x-\varepsilon)}+\sum_{i>n(x+\varepsilon)}+\sum_{n(x-\varepsilon)<i<n(x+\varepsilon)}\Big].$$

With the last three terms denoted by Σ_1, Σ_2, Σ_3 respectively, the following estimates are easily obtained

$$0\leqslant\Sigma_1\leqslant\frac{\delta}{n}\sum_{i<n(x-\varepsilon)} f_{n,i}(x)\leqslant\frac{\delta}{n}\sum_{i=1}^{n} f_{n,i}(x)\leqslant\delta,$$

$$0\leqslant\Sigma_2\leqslant\frac{\delta}{n}\sum_{i>n(x+\varepsilon)}[1-f^{\alpha-1}_{n,i}(x)]\leqslant\frac{\delta}{n}\sum_{i=1}^{n} 1=\delta,$$

$$0\leqslant\Sigma_3\leqslant\frac{1}{n}\sum_{n(x-\varepsilon)<i<n(x+\varepsilon)} 1\leqslant 2\varepsilon.$$

Hence the lemma is proved for $\alpha\geqslant 1$. It remains to consider the case $0<\alpha<1$. Since we have