DIFFERENCE METHOD FOR MULTI-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATIONS WITH WAVE OPERATOR*

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In solving physical problems the multi-dimensional nonlinear Schrödinger equations with wave operator are often obtained. Clearly, their solution requires the use of the numerical method. In this paper, we consider a difference method for the equations and prove the convergence and stability of the difference solution on the basis of prior estimates.

We consider the following periodic initial-value problem

$$\boldsymbol{u}_{tt}-\sum_{k,s=1}^{M}\frac{\partial}{\partial x_k}A_{k,s}(x)\frac{\partial \boldsymbol{u}}{\partial x_s}+\sum_{s=1}^{M}C_s(x)\frac{\partial^2\boldsymbol{u}}{\partial x_s\partial t}+R(x)\boldsymbol{u}+P(x)\boldsymbol{u}_t$$

$$+\sum_{s=1}^{M}B_s(x)\frac{\partial \boldsymbol{u}}{\partial x_s}+d(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}=\boldsymbol{f}(x,t), \tag{1}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x), \qquad \boldsymbol{u}_t|_{t=0}=\boldsymbol{u}_1(x), \tag{2}$$

$$\boldsymbol{u}(x_1,x_2,\cdots,x_m+E_m,\cdots,x_M,t)=\boldsymbol{u}(x_1,x_2,\cdots,x_m,\cdots,x_M,t), \quad 1\leqslant m\leqslant M, \tag{3}$$

where the unknown vector $\boldsymbol{u}(x,t)$ is the L-dimensional vector of the complex-valued functions, $\boldsymbol{f}(x,t)$, $\boldsymbol{u}_0(x)$ and $\boldsymbol{u}_1(x)$ are given L-dimensional vectors of the complex-valued functions, $C_s(x)$, $d(x)$ and $q(s)$ are given real functions,

$$A_{k,s}(x)=\begin{pmatrix} a^1_{k,s}(x) & & & \\ & a^2_{k,s}(x) & & \\ & & \ddots & \\ & & & a^L_{k,s}(x)\end{pmatrix}$$

is the diagonal matrix of the real-valued functions, $P(x)=(p_{l,k})_{L\times L}$, $R(x)=(r_{l,k})_{L\times L}$ and $B_s(x)=(b^{(s)}_{l,k})_{L\times L}$ are the matrixes of the complex-valued functions. The constants E_m denote periods. Because of the periodic property, the region of numerical computation is $\Omega=[0,E_1]\times[0,E_2)\times\cdots\times[0,E_m]$. In $[0,E_m]$ the step size is $h_m=\frac{E_m}{J_m}$ and the points of the net are $0, h_m, \cdots, E_m$. Let

$$\Omega_h\equiv\{x_{j_1,j_2,\cdots,j_M}|0\leqslant j_m\leqslant J_m-1\},$$

$$(f^n_{j_1,j_2,\cdots,j_M})_{x_m}=\frac{1}{h_m}(f^n_{j_1,j_2,\cdots,j_m+1,\cdots,j_M}-f^n_{j_1,j_2,\cdots,j_m,\cdots,j_M}),$$

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$$(f^n_{j_1,j_2,\cdots,j_M})_{\bar{x}_m}=\frac{1}{h_m}(f^n_{j_1,j_2,\cdots,j_m,\cdots,j_M}-f^n_{j_1,j_2,\cdots,j_m-1,\cdots,j_M}),$$

$$(f^n_{j_1,j_2,\cdots,j_M})_{\hat{x}_m}=\frac{1}{2h_m}(f^n_{j_1,j_2,\cdots,j_m+1,\cdots,j_M}-f^n_{j_1,j_2,\cdots,j_m-1,\cdots,j_M}).$$

The difference symbols for t are similar. Let the inner product

$$(\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M},\ \boldsymbol{v}^n_{j_1,j_2,\cdots,j_M})=h_1h_2\cdots h_M\sum_{l=1}^{L}\sum_{\Omega_h}u^n_{l,j_1,j_2,\cdots,j_M}\cdot\bar{v}^n_{l,j_1,j_2,\cdots,j_M}.$$

$\|f\|_{H^m}$ denotes the norm of the space $H^m(\Omega)$, $\|f\|_{L_\infty}=\operatorname{ess\,sup}_{x\in\Omega}|f(x)|$.

For the periodic initial-value problem (1)—(3), we use the following implicit scheme

$$\begin{aligned}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{t\bar{t}}&-\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k}+\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}}\\&+R_{j_1,j_2,\cdots,j_M}\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}+P_{j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}+\sum_{s=1}^{M}B_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s}\\&+d_{j_1,j_2,\cdots,j_M}q(|\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}|^2)\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}=\boldsymbol{f}^{n+1}_{j_1,j_2,\cdots,j_M},\quad(j_1,j_2,\cdots,j_M)\in\Omega_h,\end{aligned}\tag{4}$$

$$\boldsymbol{u}^0_{j_1,j_2,\cdots,j_M}=\boldsymbol{u}_0(j_1h_1,\ j_2h_2,\ \cdots,\ j_Mh_M),\quad(\boldsymbol{u}^0_{j_1,j_2,\cdots,j_M})_t=\boldsymbol{u}_1(j_1h_1,\ j_2h_2,\ \cdots,\ j_Mh_M),\tag{5}$$

$$\boldsymbol{u}(x_1,\ x_2,\ \cdots,\ x_m+E_m,\ \cdots,\ x_M,\ t)=\boldsymbol{u}(x_1,\ x_2,\ \cdots,\ x_m,\ \cdots,\ x_M,\ t),\quad 1\leqslant m\leqslant M,\tag{6}$$

where $\quad A_{k,s,j_1,j_2,\cdots,j_M}\equiv A_{k,s}\Big(j_1h_1,\ j_2h_2,\ \cdots,\ \Big(j_k+\frac{1}{2}\Big)h_k,\ \cdots,\ j_Mh_M\Big).$

Lemma 1. *Assume that*

(i) $$a^l_{k,s}(x)=a^l_{s,k}(x),\quad\sum_{k,s=1}^{M}a^l_{k,s}(x)\xi_k\bar{\xi}_s\geqslant\gamma\sum_{k=1}^{M}|\xi_k|^2,$$

for $k,\ s=1,\ 2,\ \cdots,\ M$, $1\leqslant l\leqslant L$, *where* γ *is a positive constant;*

(ii) $$d(x)\geqslant0,\ Q(s)\geqslant0,\ q'(s)\geqslant0,$$

for $s\in[0,\ \infty)$, $\int_\Omega d(x)Q(|\boldsymbol{u}_0|^2)\,dx<\infty$, *where* $Q_s=\int_0^s q(z)dz$;

(iii) $$|a^l_{k,s}(x)|\leqslant K_A,\ |b^{(s)}_{l,k}(x)|\leqslant K_B,\ |c_s(x)|\leqslant K_C,\ \left|\frac{\partial c_s(x)}{\partial x_\beta}\right|\leqslant K_C$$
$$|p_{l,k}(x)|\leqslant K_P,\ |r_{l,k}(x)|\leqslant K_R,\quad 1\leqslant s,\ \beta\leqslant M;$$

(iv) $$\boldsymbol{f}(x,\ t)\in C^0,\ \boldsymbol{u}_0(x)\in C^1,\ \boldsymbol{u}_1(x)\in C^0.$$

Then we have the estimates

$$\|\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M}\|_{L_2}\leqslant C_1,\ \|(\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|_{L_2}\leqslant C_1,\ \|(\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M})_{x_k}\|_{L_2}\leqslant C_1,\quad 1\leqslant k\leqslant M,$$

where $0\leqslant n\cdot\Delta t\leqslant T$, *the constant* C_1 *is independent of* Δt *and* h_m.

Proof. Multiplying (4) with $\overline{(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}}$ and taking the inner product, we obtain

$$\begin{aligned}&((\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{t\bar{t}},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})-\Big(\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)\\&+\Big(\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)+(\boldsymbol{R}_{j_1,j_2,\cdots,j_M}\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})\\&+(P_{j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})+\Big(\sum_{s=1}^{M}B_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)\\&+(d_{j_1,j_2,\cdots,j_M}\cdot q(|\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}|^2)\,\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})=(\boldsymbol{f}^{n+1}_{j_1,j_2,\cdots,j_M},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}).\end{aligned}\tag{7}$$