COMPUTATION OF DISCONTINUOUS SOLUTIONS OF HYPERBOLIC SYSTEMS WITH WEAKLY CONSERVATIVE DIFFERENCE SCHEMES*

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It is well known that for the computation of discontinuous solutions of hyperbolic partial differential equations, the use of conservative difference schemes has partial theoretical justification. The theorem of Lax and Wendroff in [1] states that for a conservative difference approximation of a conservative hyperbolic system $\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0$, if the difference solution converges boundedly almost everywhere, then the limit function is a weak solution of the original system of partial differential equations, and hence satisfies the Rankine Hugoniot condition. Of-course the weak solution obtained may not be the unique physically relevant solution, but under normal circumstances it will be. Now, for real practical problems the partial differential equations often have nonhomogeneous terms and the computational regions usually require coordinate transformations for simplification. Therefore we consider hyperbolic systems with coefficients which depend only on the independable variables and with nonhomogeneous terms——we call such systems weakly conservative. Computational experience over the years tells us that the use of weakly conservative difference schemes derived from the weakly conservative hyperbolic systems also yields in general the correct discontinuous solutions. The reason will be stated and proven in this note.

First of all, let us observe that the Lax and Wendroff theorem holds also for equations with nonhomogeneous terms. That is, for

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+B=0, \tag{1}$$

where U, $F(x, t, U)$ and $B(x, t, U)$ are vectors, its weak solution U satisfies

$$\iint\left(\frac{\partial W}{\partial t}\cdot U+\frac{\partial W}{\partial x}\cdot F-W\cdot B\right)dx\,dt+\int W(x, 0)\cdot U(x, 0)dx=0 \tag{2}$$

for every test function W which has continuous first derivatives and which vanishes outside some bounded region. Suppose (1) has difference approximation

$$\frac{\Delta V}{\Delta t}+\frac{\Delta G}{\Delta x}+C=0, \tag{3}$$

where Δ denotes any difference operator. From consistency we have

$$G(x, t, V(x-k\Delta x, t), \cdots, V(x+l\Delta x, t)\to G_0(x, t, V, \cdots, V)=F(x, t, V),$$
$$C(x, t, V(x-m\Delta x, t), \cdots, V(x+n\Delta x, t)\to C_0(x, t, V, \cdots, V)=B(x, t, V),$$

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here k, l, m, n are constants. For a given mesh, also denoted by Δ, a discrete solution can be obtained with (3) and then $V_\Delta(x, t)$ in the entire computational region can be defined by interpolation. With only slight modification of the proof of the Lax and Wendroff theorem in [1], we obtain the following result: if as $\Delta x, \Delta t \to 0$, $V_\Delta(x, t)$ converges boundedly almost everywhere to a function $U(x, t)$, then $U(x, t)$ is a weak solution of (1).

Now consider the coordinate transformation defined by

$$\xi = \xi(x, t), \quad \eta = \eta(x, t) \tag{4}$$

with

$$J^{-1} = \frac{\partial(\xi, \eta)}{\partial(x, t)} \neq 0, \quad J = \frac{\partial(x, t)}{\partial(\xi, \eta)} \neq 0$$

in the region under consideration. In the new variables (1) is

$$\eta_t \frac{\partial U}{\partial \eta} + \xi_t \frac{\partial U}{\partial \xi} + \eta_x \frac{\partial F}{\partial \eta} + \xi_x \frac{\partial F}{\partial \xi} + B = 0, \tag{5}$$

here η_t, ξ_t, η_x, ξ_x are considered as functions of ξ and η. It has difference approximation

$$\eta_t \frac{\Delta V}{\Delta \eta} + \xi_t \frac{\Delta V}{\Delta \xi} + \eta_x \frac{\Delta G}{\Delta \eta} + \xi_x \frac{\Delta G}{\Delta \xi} + C = 0. \tag{6}$$

Both (5) and (6) are weakly conservative, but they can be written in forms (1) and (3) respectively. Equation (5) can be written as

$$\frac{\partial \tilde{U}}{\partial \eta} + \frac{\partial \tilde{F}}{\partial \xi} + \tilde{B} = 0, \tag{7}$$

where

$$\tilde{U} = \eta_t U + \eta_x F, \quad \tilde{F} = \xi_t U + \xi_x F,$$

$$\tilde{B} = B - (\xi_t)_\xi U - (\eta_t)_\eta U - (\xi_x)_\xi F - (\eta_x)_\eta F; \tag{8}$$

and with

$$\tilde{V} = \eta_t V + \eta_x G, \quad \tilde{G} = \xi_t V + \xi_x G.$$

(6) can be written as

$$\frac{\Delta \tilde{V}}{\Delta \eta} + \frac{\Delta \tilde{G}}{\Delta \xi} + \tilde{C} = 0, \tag{9}$$

where $\tilde{C}$ includes terms $\frac{\Delta \xi_t}{\Delta \xi} V$ etc. Since (7) and (9) are of forms (1) and (3) respectively, we have: if the difference solution $\tilde{V}$ of (9), or rather $\tilde{V}$ defined by solution V of (6), converges to $\tilde{U}$, then $\tilde{U}$ is a weak solution of (7). On the ξ, η plane, $\tilde{U}$ is a weak solution of (7) if it satisfies

$$\iint\left(\frac{\partial \tilde{W}}{\partial \eta} \cdot \tilde{U} + \frac{\partial \tilde{W}}{\partial \xi} \cdot \tilde{F} - \tilde{W} \cdot \tilde{B}\right) d\xi\, d\eta + \int \tilde{W}(\xi, 0) \cdot \tilde{U}(\xi, 0) d\xi = 0 \tag{10}$$

for every test function $\tilde{W}$. Here we have assumed that $t=0$ is mapped onto $\eta=0$ and that $t>0$ corresponds to $\eta>0$, otherwise the single integral in (10) would have a minus sign in front. Let us simply call U which defines $\tilde{U}$ which satisfies (10) a weak solution of (5).

We discuss first weak solutions which are piecewise continuously differentiable in regions separated by smooth curves. The smooth parts of the solutions of (1), (5), and (7) are the same because the equations are equivalent. The discontinuity condition for (1) is