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A NOTE ON JACOBI SPECTRAL-COLLOCATION METHODS FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS WITH SMOOTH SOLUTIONS*

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Abstract

This work is concerned with spectral Jacobi-collocation methods for Volterra integral equations of the second kind with a weakly singular of the form $(t-s)^{-\alpha}$. When the underlying solutions are sufficiently smooth, the convergence analysis was carried out in [Chen & Tang, J. Comput. Appl. Math., 233 (2009), pp. 938-950]; due to technical reasons the results are restricted to $0<\mu<\frac{1}{2}$. In this work, we will improve the results to the general case $0<\mu<1$ and demonstrate that the numerical errors decay exponentially in the infinity and weighted norms when the smooth solution is involved.

Mathematics subject classification: 35Q99, 35R35, 65M12, 65M70.
Key words: Volterra integral equations, Convergence analysis, Spectral-collocation methods.

1. Introduction

We consider the linear Volterra integral equations (VIEs) of the second kind with weakly singular kernels

$$y(t)=g(t)+\int_0^t (t-s)^{-\mu}K(t,s)y(s)ds, \qquad t\in I, \tag{1.1}$$

where $I=[0,T]$, the function $g\in C(I)$, $y(t)$ is the unknown function, $\mu\in(0,1)$ and $K\in C(I\times I)$ with $K(t,t)\neq 0$ for $t\in I$. Several numerical methods have been proposed for (1.1) (see, e.g., [1,5,17,18]). For (1.1) without the singular kernel (i.e., $\mu=0$), spectral methods and the corresponding error analysis have been provided recently [19].

As the first derivatives of the solution $y(t)$ behave like $y'(t)\sim t^{-\mu}$ (see, e.g., [1]), it is difficult to employ high order numerical methods for solving (1.1). In [4], a Jacobi-collocation spectral method is developed for (1.1). To handle the non-smoothness of the underlying solutions, both function transformation and variable transformation are used to change the equation into a new Volterra integral equation defined on the standard interval $[-1,1]$. However, the function transformation (see also [5]) generally makes the resulting equations and approximations more

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complicated. We also point out a relevant recent work [8] where we consider the case with $\mu = 1/2$ (the so-called Abel integral equations) and with *non-smooth* solutions. In [8], only coordinate transformation is used but the solution transformation is not used; where in [4] both transformations are employed.

On the other hand, due to the presence of the singular factor $(t-s)^{-\mu}$ in (1.1), even with the smoothness assumption of the underlying solutions there are still difficulties in establishing a framework to obtain spectral accuracy (i.e., errors decay exponentially with the increase the degree of freedom). The study of establishing the framework was carried out in [3], but due to the technical reason the convergence results were obtained for $0 < \mu < \frac{1}{2}$ only. The main purpose of this work is to extend the results in [3] to more general values of μ, i.e., $0 < \mu < 1$. The main technical difference between this work and [3] will be pointed out at the end of Section 2. Moreover, unlike [4, 8], neither coordinate nor solution transformation will be used due to the smoothness assumption of the underlying solutions.

This paper is organized as follows. In Section 2, we outline the spectral approaches for (1.1). Some lemmas useful for establishing the convergence results will be provided in Section 3. The convergence analysis will be carried out in Section 4.

2. Jacobi-collocation Methods

Let $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ be a weight function in the usual sense, for $\alpha,\ \beta > -1$. As defined in [2, 7, 16], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1) = \{v : \ v \text{ is measurable and } ||v||_{\omega^{\alpha,\beta}} < \infty\},$$

equipped with the norm

$$||v||_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} |v(x)|^2 \omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

and the inner product

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx, \qquad \forall\ u,v \in L^2_{\omega^{\alpha,\beta}}(-1,1).$$

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^{N}$, which is the set of $(N+1)$ Jacobi Gauss points, corresponding to the weight $\omega^{-\mu,-\mu}(x)$. Let $\mathcal{P}_N$ denote the space of all polynomials of degree not exceeding N. For any $v \in C[-1,1]$, we can define the Lagrange interpolating polynomial $I_N^{\alpha,\beta}v \in \mathcal{P}_N$, satisfying

$$I_N^{\alpha,\beta}v(x_i) = v(x_i), \qquad 0 \le i \le N, \tag{2.1}$$

see, e.g., [2, 7, 16]. The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta}v(x) = \sum_{i=0}^{N} v(x_i)F_i(x),$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^{N}$.

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t = \frac{1}{2}T(1+x), \qquad s = \frac{1}{2}T(1+\tau),$$