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ERROR ESTIMATES OF FINITE ELEMENT METHODS FOR STOCHASTIC FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract

This paper studies the Galerkin finite element approximations of a class of stochastic fractional differential equations. The discretization in space is done by a standard continuous finite element method and almost optimal order error estimates are obtained. The discretization in time is achieved via the piecewise constant, discontinuous Galerkin method and a Laplace transform convolution quadrature. We give strong convergence error estimates for both semidiscrete and fully discrete schemes. The proof is based on the error estimates for the corresponding deterministic problem. Finally, the numerical example is carried out to verify the theoretical results.

Mathematics subject classification: 60H15, 65M60, 60H35, 65M12.
Key words: Stochastic fractional differential equations; Finite element method; Error estimates; Strong convergence; Convolution quadrature.

1. Introduction

In this paper we will consider the finite element approximations of the following stochastic fractional differential equations(SFDEs)

$$\begin{cases} D_t^\alpha u(t) + Au(t) = I_t^{1-\alpha}\dot{W}(t), & \alpha \in (0,1), t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{1.1}$$

The process $\{u(t)\}_{t\in[0,T]}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, takes values in a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The initial data u_0 is H-valued and $\mathcal{F}_0$-measurable random variable. The process W is a nuclear Q-Wiener process with respect to the filtration with values in some separable Hilbert space U. Let Q be a selfadjoint and positive semidefinite operator with finite trace. The operator $A : \mathcal{D}(A) \subset H \to H$ is an unbounded, densely defined, linear, selfadjoint operator with compact inverse.

Equations of the above type have many physical applications in many fields such as turbulence, heterogeneous flows and materials, viscoelasticity and electromagnetic theory [1–3], so the study of the SFDEs has recently attracted a lot of attention.

In the model (1.1), $I_t^{1-\alpha}$ is the fractional integral operator:

$$I_t^{1-\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s,$$

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and D_t^α denotes the Caputo fractional derivative of order $\alpha(0 < \alpha < 1)$ with respect to t and is defined by [4, 16]

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} u(s) \mathrm{d}s.$$

It is known that for the fractional order $\alpha = 1$, the fractional derivative D_t^α recovers the canonical first-order derivative $\frac{d}{dt}u(t)$ and thus the model (1.1) becomes the standard stochastic partial differential equation (SPDE), the numerical approximations of which have been extensively researched in the literature, see, for example, [17–22, 26–30]. However, to our best knowledge, the numerical analysis of the SFDEs is less studied, we only note that [25].

We note that there are a few papers considering the error estimates of the solution for the stochastic Volterra equations with a positive-type memory term [15, 23, 24]:

$$\mathrm{d}u + \left(A \int_0^t b(t-s) u(s) \mathrm{d}s \right) \mathrm{d}t = \mathrm{d}W, \quad u(0) = u_0.$$

This equation is closely related, but different from (1.1). The discretization is achieved via an implicit Euler scheme and a Laplace transform convolution quadrature in time, and a standard continuous finite element method in space.

Our aim is to obtain strong convergence error estimates for both semidiscrete and fully discrete schemes. The discretization in space is done by a standard continuous finite element method. And the discretization in time is achieved via the piecewise constant, discontinuous Galerkin method and a Laplace transform convolution quadrature.

The structure of this paper is as follows: In Section 2, we introduce basic notations and then give the solution representation of (1.1) by using basic properties of the Mittag-Leffler function. In Section 3, we study the space semidiscrete scheme and derive error estimates for the standard Galerkin finite element method with smooth initial data. Almost optimal order error estimates are obtained. In Section 4, by making use of the discontinuous Galerkin method and a Laplace transform convolution quadrature, we prove strong error estimates for the time semidiscrete scheme with smooth initial data. In Section 5, we gather the results from the preceding sections and give the error estimate for the fully discrete scheme. Finally, in section 6, the numerical example is carried out to verify the theoretical results.

2. Preliminaries

Throughout the paper, we use the letter C to denote a constant that may not be the same form from one occurrence to another.

Let U and H be real separable Hilbert spaces with inner product $(\cdot,\cdot)$ and norms $\|\cdot\|_U$ and $\|\cdot\|_H$. $L(U, H)$ denotes the space of bounded linear operators from U to H.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. We define $L_2(\Omega, H)$ to be the space of H-valued square integrable random variables with norm

$$\|v\|_{L_2(\Omega,H)} = \mathbf{E}(\|v\|_H^2)^{\frac{1}{2}} = \left(\int_\Omega \|v(w)\|_H^2 \mathrm{d}\mathbf{P}(w) \right)^{\frac{1}{2}},$$

where $\mathbf{E}$ stands for expected value. Let Q be a selfadjoint, positive semidefinite operator, with $Tr(Q) < \infty$, where $Tr(Q)$ denotes the trace of Q. The stochastic process $W(t)$ is a U-valued Q-Wiener process on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Furthermore, $W(t)$ has the orthogonal