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A GENERAL HIGH-ORDER MULTI-DOMAIN HYBRID DG/WENO-FD METHOD FOR HYPERBOLIC CONSERVATION LAWS*

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Abstract

In this paper, a general high-order multi-domain hybrid DG/WENO-FD method, which couples a p^{th}-order ($p \geq 3$) DG method and a q^{th}-order ($q \geq 3$) WENO-FD scheme, is developed. There are two possible coupling approaches at the domain interface, one is non-conservative, the other is conservative. The non-conservative coupling approach can preserve optimal order of accuracy and the local conservative error is proved to be upmost third order. As for the conservative coupling approach, accuracy analysis shows the forced conservation strategy at the coupling interface deteriorates the accuracy locally to first-order accuracy at the 'coupling cell'. A numerical experiments of numerical stability is also presented for the non-conservative and conservative coupling approaches. Several numerical results are presented to verify the theoretical analysis results and demonstrate the performance of the hybrid DG/WENO-FD solver.

Mathematics subject classification: 65M60, 65M99, 35L65
Key words: Discontinuous Galerkin method, Weighted essentially nonoscillatory scheme, Hybrid methods, high-order scheme.

1. Introduction

In recent years, high-order methods with low numerical diffusion and dispersion errors have been extensively studied and widely used for resolving complex fluid structures and capturing vortex evolution. Many kinds of high-order methods have been developed over the past two decades, especially high-order finite difference (FD) type methods, such as weighted essentially non-oscillatory (WENO) schemes [10, 11], high-order finite volume (FV) type methods [12, 29, 30] and discontinuous Galerkin (DG) type methods, which includes traditional Runge-Kutta discontinuous Galerkin (RKDG) methods [1, 2], spectral volume (SV) methods [19, 20], spectral difference (SD) method [14], correction procedure via reconstruction (CPR) schemes [13, 21] and so on. FD type methods are generally considered as highly efficient and easily achieve high-order accuracy on structured grids. FV type methods, compared to the FD type methods, have flexibility in handling almost arbitrary grid with reasonable computational cost, however, they usually are not compact when extends to high order accuracy. DG type methods can treat complex geometries through a compact stencil as each cell only communicates with its immediate face-neighbors through approximate Riemann solvers. However, they are generally more time consuming than the FV type methods.

As all of those high-order methods mentioned above have their advantages and disadvantages, several hybrid methods have been developed in order to locally take the advantages of

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different types of methods above. There are mainly two kinds of hybrid approaches in literature, one is based on local polynomial reconstruction, the other is based on computational domain decomposition.

Based on local polynomial reconstruction, one of the most typical and successful implementation is WENO/Hermite WENO type limiter for DG methods [4–7]. The basic idea of WENO/Hermite WENO limiter is to use the local polynomial reconstruction of WENO type to replace the original reconstruction approach of DG methods for high-order degrees of freedom at the 'troubled cells'. Such hybrid approach preserves essentially non-oscillatory property for those cells near the area of discontinuities. In fact, the strategy can be applied for all the cells rather than only for those 'troubled cells'. Luo et al. [3, 8, 9] adopted this strategy and developed a kind of reconstructed DG methods (rDG) which reconstructs high-order degrees of freedom using a least-squares technique in 2D and a hierarchial WENO reconstruction in 3D. This hybrid approach combines the efficiency of reconstruction methods widely used in FV type schemes and the accuracy and robustness of DG methods. Similarly, based on local polynomial reconstruction, Dumbser et al. [15] presented a unified framework for constructing one step finite volume and discontinuous Galerkin schemes on unstructured meshes, resulting in a class of PnPm schemes. Zhang et al. [16–18] introduced a hybrid DG/FV method. In their hybrid DG/FV method, a DG method based on Taylor basis functions was adopted to compute the low-order degrees of freedom. A high order finite volume method is then used to reconstruct high-order derivatives with the known low-order derivatives.

Based on computational domain decomposition, a multi-domain hybrid spectral-WENO method was introduced by Costa et al. [22] for hyperbolic conservation laws. The hybrid spectral-WENO method conjugates the non-oscillatory properties of high-order WENO schemes and the high efficiency and accuracy of spectral methods in a multi-domain approach. Recently, Shahbazi et al. [23] introduced a multi-domain Fourier-continuation/WENO hybrid solver for conservation laws. In the area of CAA, Utzmann et al. [26,28] and Léger et al. [27] developed a coupled DG/FD method for computational aeroacoustics. The coupled DG/FD solver approximates the solution in the close neighborhood of complex obstacles with an unstructured grids and computes the rest of the field on structured grids in order to alleviate computational cost.

In our previous work [31,32], we introduced a class of multi-domain hybrid DG and WENO-FD method based on a third-order DG method and a fifth-order WENO-FD scheme for the purpose of saving computational cost and treating complex geometry. We found that the conservative coupling approach deteriorates the accuracy seriously and only the non-conservative coupling approach can preserve third-order accuracy. Thus, a special treatment was developed in our previous work: the non-conservative coupling approach is employed when the solution is smooth enough and it is replaced by the conservative coupling approach when there are possible discontinuities passing through the interface.

In this paper, as a direct extension of our previous work, two main issues will be addressed: one is whether the hybrid method can be extended to arbitrary high-order accuracy, the other is whether the hybrid method can preserve numerical stability with different choices of numerical fluxes at coupling interface. Firstly, we extend the previous third-order hybrid DG/WENO-FD method to a more general situation which couples a p^{th}-order ($p \geq 3$) DG method with a q^{th}-order ($q \geq 3$) WENO-FD scheme. We present a general analysis of accuracy and conservative error for this general high-order hybrid DG/WENO-FD method. From the theoretical analysis, as we will see later, the conservative coupling approach of a p^{th}-order DG method and a q^{th}-order WENO-FD scheme is only of first-order accuracy locally at the 'coupling cell' and the