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A NUMERICAL METHOD FOR SOLVING NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF FREDHOLM TYPE*

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Abstract

The paper deals with a numerical method for solving nonlinear integro-parabolic problems of Fredholm type. A monotone iterative method, based on the method of upper and lower solutions, is constructed. This iterative method yields two sequences which converge monotonically from above and below, respectively, to a solution of a nonlinear difference scheme. This monotone convergence leads to an existence-uniqueness theorem. An analysis of convergence rates of the monotone iterative method is given. Some basic techniques for construction of initial upper and lower solutions are given, and numerical experiments with two test problems are presented.

Mathematics subject classification: 65M06, 65N06, 65N22, 65R20
Key words: Nonlinear integro-parabolic equations of Fredholm type, Nonlinear difference schemes, Monotone iterative methods, The method of upper and lower solutions.

1. Introduction

Integro-differential equations of Fredholm type arise from various fields of applied sciences (see [4] for details). The purpose of this paper is to construct a numerical method for solving nonlinear integro-parabolic equations of Fredholm type in the form

$$\begin{aligned} &u_t - Lu + f(x,t,u) + \int_\omega g_0(x,s,t,u(s,t))ds = 0, \quad (x,t) \in \omega \times (0,T], \\ &u(x,t) = h(x,t), \quad (x,t) \in \partial\omega \times (0,T], \\ &u(x,0) = \psi(x), \quad x \in \overline{\omega}, \end{aligned} \tag{1.1}$$

where ω is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1,2,\dots$) with boundary $\partial\omega$. The linear differential operator L is given by

$$Lu = \sum_{\nu=1}^{\kappa} \frac{\partial}{\partial x_\nu}\left(D\frac{\partial u}{\partial x_\nu}\right) + \sum_{\nu=1}^{\kappa} v_\nu \frac{\partial u}{\partial x_\nu},$$

where $D = D(x,t) > 0$ and $v_\nu = v_\nu(x,t)$, $\nu = 1,\dots,\kappa$, are the diffusion and convection coefficients, T is an arbitrary positive constant. The functions D, v_ν, $\nu = 1,\dots,\kappa$, f, g_0, h and ψ are smooth in their respective domains.

To discretize problem (1.1), we use the implicit scheme for parabolic equations and approximate (1.1) by a nonlinear difference scheme. The purpose of this paper is to develop a

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monotone iterative method for solving the nonlinear difference scheme, including the existence and uniqueness of a discrete solution, and error estimates of the iterative method. Our iterative method is based on the method of upper and lower solutions and associated monotone iterates. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the nonlinear difference scheme.

Monotone iterative schemes for solving nonlinear parabolic equations were used in [1–3,5,6,8, 11]. In [7], a monotone iterative method for solving one dimensional nonlinear integro-parabolic equations of Fredholm type is presented. In [7], the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were not given. In this paper, we investigate the monotone iterative method in the case when on each time level nonlinear difference schemes are solved inexactly, and give an analysis of convergence rates of the monotone iterative method.

The plan of the paper as follows. In Section 2, we formulate a nonlinear difference scheme for the numerical solution of (1.1) by the implicit method for parabolic equations. A monotone iterative method for the nonlinear difference scheme is given in Section 3, where the nonlinear function g_0 in (1.1) is considered nondecreasing in u. Existence and uniqueness of the solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Convergence of the nonlinear difference scheme to the nonlinear integro-parabolic problem (1.1) is established, and an error estimate is obtained. In Section 4, some basic techniques for construction of initial upper and lower solutions are given, and numerical experiments with two test problems are presented.

2. The Nonlinear Difference Scheme

We introduce meshes $\overline{\omega}^h$, $\overline{\omega}^\tau$ on the domains $\overline{\omega}$ and $[0,T]$, respectively. The integral in (1.1) is approximated by the finite sum g based on a composite Newton-Cotes quadrature rule (see [10] for details)

$$g(p,t_k,U)=\sum_{l=1}^{N} b_l g_0(p,p_l,t_k,U(p_l,t_k)), \quad (p,t_k)\in\overline{\omega}^h\times\omega^\tau, \quad \omega^\tau\equiv\overline{\omega}^\tau\setminus\{0\},$$

where b_l, $l=1,\dots,N$, are nonnegative weights, and N is the number of mesh points $p\in\overline{\omega}^h$. By using the implicit method for parabolic equations, we approximate the integro-parabolic differential equation in (1.1) by the difference scheme

$$\begin{aligned}&\mathcal{L}U(p,t_k)+f(p,t_k,U)+g(p,t_k,U)-\tau_k^{-1}U(p,t_{k-1})=0, \quad (p,t_k)\in\omega^h\times\omega^\tau \qquad (2.1)\\ &\mathcal{L}U(p,t_k)=\mathcal{L}^hU(p,t_k)+\tau_k^{-1}U(p,t_k),\end{aligned}$$

where $\mathcal{L}$ is an approximation of the differential operator L from (1.1). When no confusion arises, we write $f(p,t_k,U(p,t_k))=f(p,t_k,U)$, $g(p,t_k,U(p,t_k))=g(p,t_k,U)$. The boundary and initial conditions are approximated by

$$U(p,t_k)=h(p,t_k), \quad (p,t_k)\in\partial\omega^h\times\omega^\tau, \quad U(p,0)=\psi(p), \quad p\in\overline{\omega}^h,$$