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POSITIVE DEFINITE AND SEMI-DEFINITE SPLITTING METHODS FOR NON-HERMITIAN POSITIVE DEFINITE LINEAR SYSTEMS*

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Abstract

In this paper, we further generalize the technique for constructing the normal (or positive definite) and skew-Hermitian splitting iteration method for solving large sparse non-Hermitian positive definite system of linear equations. By introducing a new splitting, we establish a class of efficient iteration methods, called positive definite and semi-definite splitting (PPS) methods, and prove that the sequence produced by the PPS method converges unconditionally to the unique solution of the system. Moreover, we propose two kinds of typical practical choices of the PPS method and study the upper bound of the spectral radius of the iteration matrix. In addition, we show the optimal parameters such that the spectral radius achieves the minimum under certain conditions. Finally, some numerical examples are given to demonstrate the effectiveness of the considered methods.

Mathematics subject classification: 65F10, 65F30, 65F50.
Key words: Linear systems, Splitting method, Non-Hermitian matrix, Positive definite matrix, Positive semi-definite matrix, Convergence analysis.

1. Introduction

Many problems in scientific computing give rise to a system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n}, \quad \textit{and} \;\; x,\; b \in \mathbb{C}^n, \tag{1.1}$$

with A being a large sparse non-Hermitian but positive definite matrix.

We call a matrix B positive definite (or positive semi-definite), if $B + B^*$ is Hermitian positive definite (or positive semi-definite), i.e., for all $0 \neq x \in \mathbb{C}^n$, $x^*(B + B^*)x > 0$ (or $x^*(B + B^*)x \geq 0$), where B^* denotes the complex conjugate transpose of the B. Let $D = \mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ be the diagonal part of A and $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$. Since the coefficient matrix A is positive definite, we have $e_i^*(A + A^*)e_i = a_{ii} + \overline{a_{ii}} > 0$. This shows that D is positive definite.

The linear system has many important practical applications, such as diffuse optical tomography, molecular scattering, lattice quantum chromodynamics (see, e.g., [1,7,8,22,24,38,39,41]). Many researchers have been devoted themselves to the numerical solution of (1.1) (see e.g., [2–4, 10, 11, 18, 21, 25, 27, 28, 36, 37, 40, 42, 45–47] and the references therein) and proposed kinds of available iteration methods for solving the system (1.1), in which splitting iteration methods (see e.g., [9,13–17,19,29–31,35,44]) and Krylov subspace methods (see e.g., [5,20,23,26,32,43])

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attract a lot of attention. In [16], the authors presented a Hermitian and skew-Hermitian splitting (HSS) iteration method for solving (1.1) and showed that the HSS method converges unconditionally to the unique solution of the system. Then many researchers focused on the HSS method and proposed kinds of iterations method based on Hermitian and skew-Hermitian splitting (see e.g., [6, 9, 12, 13, 17]). In recent years, other kinds of splitting iteration methods have also been studied (see e.g., [14,15,33–35,44]). Normal and skew-Hermitian splitting (NSS) iteration methods for solving large sparse non-Hermitian positive definite linear system was studied in [15]. Based on block triangular and skew-Hermitian splitting, a class of iteration methods for solving positive-definite linear systems was established in [14]. Krukier et. al proposed the generalized skew-Hermitian triangular splitting iteration methods to solve (1.1) and applied the methods to solve the saddle-point linear systems (see [34]). In this work, we further generalize the technique for constructing the normal (or positive definite) and skew-Hermitian splitting iteration method for solving (1.1).

Throughout this paper, we use the following notations: $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices and $\mathbb{C}^m = \mathbb{C}^{m\times 1}$. We use C and R to denote the set of complex numbers and real numbers, respectively. For any $a \in C$, we write $Re(a)$ and $Im(a)$ to denote the real and imaginary parts of a. For $B \in \mathbb{C}^{n\times n}$, we write B^{-1}, $\|B\|_2$, $\Lambda(B)$ and $\rho(B)$ to denote the the inverse, 2-norm, the spectrum and the spectral radius of the matrix B, respectively. I denotes the identity matrix of size implied by context. $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit.

The organization of this paper is as follows. In Section 2, we present the positive definite and semi-definite splitting methods for solving non-hermitian positive definite linear systems and study the convergence properties of the PPS iteration. In Section 3, we establish two kinds of typical practical choices of the PPS method and study the upper bound of the spectral radius of iteration matrix. Numerical experiments are presented in Section 4 to show the effectiveness of our methods.

2. The Positive Definite and Semi-definite Splitting Method

In this section, we study efficient iterative methods for solving (1.1) based on the positive definite and semi-definite splitting (PPS for short) of the coefficient matrix A, and establish the convergence analysis of the new methods. For this purpose, we split A into positive-definite and positive semi-definite parts as follows:

$$A = M + N, \tag{2.1}$$

where M is a positive-definite matrix and N is a positive semi-definite matrix. Then it is easy to see that

$$A = \alpha I + M - (\alpha I - N) = \alpha I + N - (\alpha I - M).$$

This implies that the system (1.1) can be reformulated equivalently as:

$$(\alpha I + M)x = (\alpha I - N)x + b,$$

or

$$(\alpha I + N)x = (\alpha I - M)x + b.$$

By the two above fixed point equations, we can get the following iteration method:

$$\begin{cases} (\alpha I + M)x^{(k+\frac{1}{2})} = (\alpha I - N)x^{(k)} + b, \\ (\alpha I + N)x^{(k+1)} = (\alpha I - M)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{2.2}$$