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Complex Oscillation of Differential Polynomials Generated by Meromorphic Solutions of [p,q] Order to Complex Non-homogeneous Linear Differential Equations

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Abstract. In this article we study the complex oscillation of differential polynomials generated by meromorphic solutions of the non-homogeneous linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F,$$

where $A_i(z)$ $(i=0,1,\cdots,k-1)$ and F are meromorphic functions of finite $[p,q]$-order in the complex plane.

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Key words: Non-homogeneous linear differential equations, differential polynomials, meromorphic solutions, $[p,q]$-order.

1 Introduction and preliminaries

In this paper, we assume that the reader knows the standard notations and the fundamental results of the Nevanlinna's value distribution theory of meromorphic functions (see [10] , [15], [22]). Throughout this paper, we assume that a meromorphic function is meromorphic in the whole complex plane $\mathbb{C}$. Let us define inductively for $r\in\mathbb{R}$, $\exp_1 r:=e^r$ and

$$\exp_{p+1}r:=\exp\left(\exp_p r\right),\ p\in\mathbb{N}.$$

We also define for all r sufficiently large $\log_1 r:=\log r$ and

$$\log_{p+1}r:=\log\left(\log_p r\right),\ p\in\mathbb{N}.$$

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Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$. In [12], [13], Juneja-Kapoor-Bajpai investigated some properties of growth of entire functions of $[p,q]$-order. In [21], in order to keep accordance with the general definitions of entire function $f(z)$ of iterated $p-$order [14], [15], Liu-Tu-Shi gave a minor modification to the original definition of $[p,q]-$order given in [12], [13]. With this new concept of $[p,q]-$order, the $[p,q]-$order of solutions of complex linear differential equations was investigated (see e.g. [2-4], [6], [11], [20], [21], [23]).

Now we introduce the definitions of the $[p,q]-$order as follows.

Definition 1.1. ([20, 21]) *Let $p \geq q \geq 1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p,q]$-order of $f(z)$ is defined by*

$$\rho_{[p,q]}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log_q r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. For $p=1$, this notation is called order and for $p=2$ hyper-order. It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. By Definition 1.1, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$ usual order, $\rho_{[2,1]}(f) = \rho_2(f)$ hyper-order and $\rho_{[p,1]}(f) = \rho_p(f)$ iterated $p-$order.

Definition 1.2. ([11]) *Let $p \geq q \geq 1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the lower $[p,q]$-order of $f(z)$ is defined by*

$$\mu_{[p,q]}(f) = \liminf_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

Remark 1.1. ([20]) If $f(z)$ is a meromorphic function satisfying $0 < \rho_{[p,q]}(f) < \infty$, then

(i) $\rho_{[p-n,q]}(f) = \infty$ $(n<p)$, $\rho_{[p,q-n]}(f) = 0$ $(n<q)$, $\rho_{[p+n,q+n]}(f) = 1$ $(n<p)$ for $n = 1,2,\cdots$

(ii) If $[p_1,q_1]$ is any pair of integers satisfying $q_1 = p_1 + q - p$ and $p_1 < p$, then $\rho_{[p_1,q_1]}(f) = 0$ if $0 < \rho_{[p,q]}(f) < 1$ and $\rho_{[p_1,q_1]}(f) = \infty$ if $1 < \rho_{[p,q]}(f) < \infty$.

(iii) $\rho_{[p_1,q_1]}(f) = \infty$ for $q_1 - p_1 > q - p$ and $\rho_{[p_1,q_1]}(f) = 0$ for $q_1 - p_1 < q - p$.

Definition 1.3. ([20]) *A transcendental meromorphic function $f(z)$ is said to have index-pair $[p,q]$ if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.*

Remark 1.2. ([20]) Suppose that f_1 is a meromorphic function of $[p,q]$-order ρ_1 and f_2 is a meromorphic function of $[p_1,q_1]$-order ρ_2, let $p \leq p_1$. We can easily deduce the result about their comparative growth:

(i) If $p_1 - p > q_1 - q$, then the growth of f_1 is slower than the growth of f_2.