Journal of Computational Mathematics
Vol.33, No.5, 2015, 517–532.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1505-m2014-0101

# THE FACTORIZATION METHOD FOR AN OPEN ARC*

Qinghua Wu and Guozheng Yan
*School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China*
*Email: wqhyimo@yeah.net, yangz@mail.ccnu.edu.cn*

**Abstract**

We consider the inverse scattering problem of determining the shape of a thin dielectric infinite cylinder having an open arc as cross section. Assuming that the electric field is polarized in the TM mode, this leads to a mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc in $R^2$. We suppose that the arc has mixed Dirichlet–impedance boundary condition, and try to recover the shape of the arc through the far field pattern by using the factorization method. However, we are not able to apply the basic theorem introduced by Kirsch to treat the far field operator $F$, and some auxiliary operators have to be considered. The theoretical validation of the factorization method to our problem is given in this paper, and some numerical results are presented to show the viability of our method.

*Mathematics subject classification:* 35Q65, 35C15, 78A45.
*Key words:* Factorization method; inverse scattering problem; crack scattering.

## 1. Introduction

In this paper, we consider the inverse scattering problem of determining the shape of a thin dielectric infinite cylinder having an open arc as cross section. To this end, we solve a mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc in $R^2$, and we focus on the inverse scattering problem, that is, we try to use the factorization method to retrieve the shape and location of the open arc in $R^2$.

In [15], Kress considered a sound-soft crack (arc in $R^2$) scattering problem and used an integral equation method to transform the scattering problem in the unbounded domain into a boundary integral equation. In [24, 25], Mönch studied the numerical solution of the direct scattering problem and the inverse scattering problem from an open arc with sound-hard boundary condition. In 2000, Kress's work was continued by Kirsch and Ritter in [12] who used the linear sampling method to reconstruct the shape of the crack from the knowledge of the far field pattern, and in the same year these results were generalized to the scattering problem with cracks for Maxwell equations by Ammari, et al. [3]. In [6], Cakoni and Colton discussed the direct and inverse scattering problems for cracks (possibly) coated on one side by a material. Extending to the impedance problem, Lee [13] considered the direct and inverse scattering problem for an impedance crack. In [10], Hassen et al. discussed a kind of scattering problem for impedance cracks and reconstructed the shape of the crack by using the linear sampling method. More recently, Boukari and Haddar reconstructed the shape of cracks with impedance boundary conditions by using the factorization method in [4]. More related results can be found in [1, 2, 7, 14, 16, 17] and the reference therein.

* Received October 17, 2014 / Revised version received March 23, 2015 / Accepted May 4, 2015 /
Published online September 18, 2015 /

The factorization method gives a sample and fast algorithm by using the behavior of an indicator function, which was introduced by Kirsch and Grinberg in [18], and has been applied extensively in the inverse scattering problems, see, e.g., [9-19, 22]. In this paper, we try to use this method to retrieve an open arc with mixed Dirichlet–impedance boundary condition.

However, in the case of the mixed Dirichlet-impedance boundary condition, we are not able to apply the basic theorem (Theorem 2.15 in [18]) introduced by Kirsch to treat the far field operator $F$. This is because the decomposition operator $M$ of the far field operator $F$ fails to satisfy the second condition of the basis theorem. To handle this, we introduce a modified operator $F_1$ to replace $F$. The new operator $F_1$ can be decomposed so that the factorization method can be used. The main challenge of this paper is to factorize the auxiliary operator suitably, and show some key properties to the related operators.

The paper is organized as follows. In the next section, we will formulate the direct scattering problem. The mathematical basis of the factorization method applied to treat the inverse scattering problem is given in Section 3. In Section 4, some numerical results are presented to show the viability of our method.

## 2. The Direct Scattering Problem

In this paper we consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open arc in $R^2$ as cross section. Assuming that the electric field is polarized in the TM mode, this leads to a (possibly) mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc.

Let $u^i = e^{ikx\cdot d}$ be the incident plane wave, the positive wave number $k = \omega/c$ with frequency $\omega$, sound speed $c$ and incident direction $d \in S_1$. Here and in the following, $S_1$ denotes the unit circle in $R^2$.

Then the total field $U$ of the scattering problem from an open arc with a mixed Dirichlet-impedance boundary condition satisfies the following boundary value problem

$$\begin{cases} \Delta U + k^2 U = 0 & \text{in} \quad R^2\backslash\overline{\Gamma} \\ U_- = 0 & \text{on} \quad \Gamma \\ (\frac{\partial}{\partial\nu} + \lambda)U_+ = 0 & \text{on} \quad \Gamma \end{cases} \tag{2.1}$$

where the total field $U = u^i + u^s$ is the combination of the incident wave $u^i$ and the scattering field $u^s$ which satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2.2}$$

with $r = |x|$, and (2.2) holds uniformly in all directions $\hat{x} = x/|x|$.

It is known that $u^s$ has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}}\left\{u^\infty(\hat{x}, d) + O(1/r)\right\} \tag{2.3}$$

as $r = |x| \to \infty$. The function $u^\infty(\hat{x}, d)$ defined on the unit circle $S_1$ is known as the far field pattern with $\hat{x}$ denoting the observation direction.

The inverse scattering problem we consider in this paper is to determine the shape of the open arc $\Gamma$ from the knowledge of the far field pattern $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S_1$ with the given wave number $k$.