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MULTIGRID METHOD FOR FLUID DYNAMIC PROBLEMS*

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Abstract

This paper covers the review and some aspects of using Multigrid method for fluid dynamics problems. The main development stages of multigrid technics are presented. Some approaches for solving Navier-Stokes equations and convection- diffusion problems are considered.

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Key words: Navier-Stokes equation, Convection-diffusion, Finite element method, Multigrid method.

1. Introduction

Fluid dynamics and transport phenomena, such as heat and mass transfer, play an important role in human life. Gases and liquids surround us, flow inside our bodies have a profound influence on the environment in which we live. Fluid flows produce winds, rains, floods, and hurricanes. Convection and diffusion are responsible for temperature fluctuations and transport of pollutants in air, water or soil.

The ability to understand, predict, and control transport phenomena is essential for many industrial applications, such as aerodynamic shape design, oil recovery from an underground reservoir, or multiphase/multicomponent flows in furnaces, heat exchangers, and chemical reactors. This ability offers substantial economic benefits and contributes to human well-being. Heating, air conditioning, and weather forecast have become an integral part of our everyday life. Most people take such things for granted and hardly ever think about the physics and mathematics behind them [1]. In physics fluid dynamics is a subdiscipline of fluid mechanics that deals with fluid flow-the natural science of fluids (liquids and gases) in motion. It has several subdisciplines itself, including aerodynamics (the study of air and other gases in motion) and hydrodynamics (the study of liquids in motion).

Before the twentieth century, hydrodynamics was synonymous with fluid dynamics. This is still reflected in names of some fluid dynamics topics, like magnetohydrodynamics and hydrodynamic stability, both of which can also be applied to gases.

Fluid dynamics has a wide range of applications, including calculating forces and moments on aircraft, determining the mass flow rate of petroleum through pipelines, predicting weather patterns, understanding nebulae in interstellar space. Some of its principles are even used in traffic engineering, where traffic is treated as a continuous fluid.

Computational fluid dynamics, usually abbreviated as CFD, is a branch of fluid mechanics that uses numerical methods and algorithms to solve and analyze problems that involve fluid flows. The fundamental basis of almost all CFD problems are the Navier-Stokes equations,

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which define any single-phase (gas or liquid, but not both) fluid flow. These equations can be simplified by removing terms describing viscous actions to yield the Euler equations. Further simplification, by removing terms describing vorticity yields the full potential equations. Finally, for small perturbations in subsonic and supersonic flows (not transonic or hypersonic) these equations can be linearized to yield the linearized potential equations.

The other most common equation in the computational fluid dynamics field is the convection-diffusion equation. Mathematical models that involve a combination of convective and diffusive processes are among the most widespread in all the sciences. Research of these processes is especially important and difficult when convection is dominant. At the same time, convection-diffusion equations are used as tests in researching iterative methods for solving systems of strongly nonsymmetric linear equations.

During the simulation of some physical phenomena, in the CFD, the solution of large linear systems is usually required. With the ongoing increase of the complexity of the problems to treat, the solution phase may be very costly. It is not sufficient to use the latest technology of computers. An effort should be put into algorithms for solving such systems. For such systems, involving several millions of degrees of freedoms, direct methods [2] are not convenient, and iterative methods [3] usually suffer from a low convergence speed. Hybrid methods, like domain decomposition methods [4], can be considered. These methods consist to split the global system to solve into multiple sub-systems, each subsystem being solved independently by sharing information along so called interface conditions between neighboring sub-systems. These interface conditions can be optimized for the performance of the algorithm [5]. Unfortunately, these methods might suffer from convergence problems, and suitable preconditioning techniques lead to an additional computational cost.

Multigrid methods (MGM) are known to be a viable alternative to the previous solution strategies especially for elliptic dominated problems [6]. They are the fastest numerical methods for solving boundary value problems [7]. Multigrid methods were the first to overcome the complexity barrier connected with that the amount of work does not remain proportional to the number of unknowns. The starting point of the multigrid and indeed also its ultimate upshot is the following "golden" rule: The amount of computational work should be proportional to the amount of real physical changes in the computed system.

The field of multigrid methods has become too large to review in a single article. Therefore, in this paper, we restrict our attention to the class of problems which is actual one for fluid dynamics: Navier-Stokes equations and convection- diffusion problems.

2. Multigrid Method: Main Development Stages

First working multigrid method was developed and analyzed by Fedorenko [8] for the Laplace equation on the unit square. Bachvalov [9] considered the theoretically much more complex case of variable coefficients.

The main observation of multigrid techniques is based on a Fourier analysis of the residual (or error) vector of a sequence of iterates that are generated by a scheme such as Jacobi or Gauss-Seidel (for instance). This means that these residual vectors are analyzed in the eigen-basis associated with the iteration matrix M - assuming that M has a complete set of eigenvectors. In the case of Jacobi, the observation is that the components associated with the largest eigenvalues (in the original matrix) will decrease rapidly. However, those associated with the smallest eigenvalues will converge much more slowly. As a result after a few steps, the