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SPECTRAL AND SPECTRAL ELEMENT METHODS FOR HIGH ORDER PROBLEMS WITH MIXED BOUNDARY CONDITIONS*

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Abstract

In this paper, we investigate numerical methods for high order differential equations. We propose new spectral and spectral element methods for high order problems with mixed inhomogeneous boundary conditions, and prove their spectral accuracy by using the recent results on the Jacobi quasi-orthogonal approximation. Numerical results demonstrate the high accuracy of suggested algorithm, which also works well even for oscillating solutions.

Mathematics subject classification: 65L60, 65M70.
Key words: Spectral and spectral element methods, High order problems with mixed inhomogeneous boundary conditions.

1. Introduction

The Legendre and Chebyshev spectral methods have been used successfully for numerical solutions of differential equations, see, e.g., [3, 6, 7, 11, 12, 24] and the references therein. Some authors also developed the Jacobi spectral method of singular differential equations, see, e.g., [13, 14, 20]. Guo et al. [17, 18] proposed the generalized Jacobi spectral method enlarging the applications. We considered second order problems mostly. However, it is also important to deal with high order problems arising in science and engineering, see, e.g., [2,4,5,8,15,22,25-27] and the references therein. Recently, Guo et al. [19] developed the generalized Jacobi quasi-orthogonal approximation, which generalize the Legendre quasi-orthogonal approximation given in [16, 21, 23]. In particular, it leads to the probability of producing new spectral and spectral element methods for high order problems with various mixed boundary conditions.

In this work, we investigate new numerical methods for high order differential equations, by using the recent results on the Jacobi quasi-orthogonal approximation. The next section is for preliminaries. In Section 3, we propose spectral element method for high and even order problems with mixed inhomogeneous Dirichlet-Neumann boundary conditions, and prove its spectral accuracy. We also provide the spectral element method with essential imposition of mixed boundary conditions. In Section 4, we consider spectral method for high and odd order

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problem and prove its spectral accuracy in space. In Section 5, we present some numerical results demonstrating the effectiveness of suggested algorithm, which also works well even for oscillating solutions. The final section is for concluding remarks. Although we only considered two model problems in this paper, the idea and techniques developed in this work open a new goal for designing and analyzing spectral and spectral element methods of many other high order problems with various boundary conditions.

2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. For any integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ as usual, with the inner product $(u,v)_{r,\chi}$, the semi-norm $|v|_{r,\chi}$ and the norm $\|v\|_{r,\chi}$. In particular, we denote the inner product and the norm of $L^2_\chi(\Lambda)$ by $(u,v)_\chi$ and $\|v\|_\chi$, respectively. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in Λ. We omit the subscript χ in notations, whenever $\chi(x) \equiv 1$.

Let $\chi^{(\sigma,\lambda)}(x) = (1-x)^\sigma(1+x)^\lambda$, $\sigma, \lambda > -1$, and $J^{(\sigma,\lambda)}_l(x)$ be the Jacobi polynomials of degree l. For any integers $m, n \geq 1$, we set

$$Y^{(m,n)}_l(x) = (1-x)^m(1+x)^n J^{(m,n)}_{l-m-n}(x), \qquad l \geq m+n.$$

The set of all polynomials $Y^{(m,n)}_l(x)$ is a complete $L^2_{\chi^{(-m,-n)}}(\Lambda)$-orthogonal system, namely (see (2.9) of [18]),

$$\int_\Lambda Y^{(m,n)}_l(x) Y^{(m,n)}_{l'}(x) \chi^{(-m,-n)}(x) dx = \gamma^{(m,n)}_l \delta_{l,l'}, \tag{2.1}$$

where $\delta_{l,l'}$ is the Kronecker symbol, and

$$\gamma^{(m,n)}_l = \frac{2^{m+n+1}\Gamma(l-m+1)\Gamma(l-n+1)}{(2l-m-n+1)\Gamma(l+1)\Gamma(l-m-n+1)}, \qquad l \geq m+n.$$

For any $v \in L^2_{\chi^{(-m,-n)}}(\Lambda)$, we have

$$v(x) = \sum_{l=m+n}^{\infty} \hat{v}^{(m,n)}_l Y^{(m,n)}_l(x), \tag{2.2}$$

where

$$\hat{v}^{(m,n)}_l = \frac{1}{\gamma^{(m,n)}_l} \int_\Lambda v(x) Y^{(m,n)}_l(x) \chi^{(-m,-n)}(x) dx.$$

For any positive integer N, $\mathcal{P}_N(\Lambda)$ stands for the set of all algebraic polynomials of degree at most N, and

$$\mathcal{Q}^{(m,n)}_N(\Lambda) = \mathrm{span}\{Y^{(m,n)}_l(x),\ m+n \leq l \leq N\}.$$

The projection $P_{N,m,n} : L^2_{\chi^{(-m,-n)}}(\Lambda) \to \mathcal{Q}^{(m,n)}_N(\Lambda)$ is defined by

$$(P_{N,m,n}v - v, \phi)_{\chi^{(-m,-n)}} = 0, \qquad \forall \phi \in \mathcal{Q}^{(m,n)}_N(\Lambda). \tag{2.3}$$

For numerical solutions of high order differential equations, we need other orthogonal projections. For this purpose, we introduce the space

$$H^r_{m,n,A}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{H^r_{m,n,A}} < \infty\},$$