Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **xx**, No. x, pp. 1-18

DOI: 10.4208/aamm.OA-2022-0108
xxx 2024

A Primal-Dual Discontinuous Galerkin Finite Element Method for Ill-Posed Elliptic Cauchy Problems

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Received 16 April 2022; Accepted (in revised version) 17 February 2023

Abstract. We present a primal-dual discontinuous Galerkin finite element method for a type of ill-posed elliptic Cauchy problem. It is shown that the discrete problem attains a unique solution, if the solution of the ill-posed elliptic Cauchy problems is unique. An optimal error estimate is obtained in a H^1-like norm. Numerical experiments are provided to demonstrate the efficiency of the proposed method.

AMS subject classifications: 65N15, 65N30

Key words: The ill-posed elliptic problem, discontinuous Galerkin method, primal-dual scheme, optimal error estimate.

1 Introduction

In this paper, we consider the following ill-posed elliptic Cauchy problem

$$\begin{cases} -\nabla\cdot(a\nabla u)=f & \text{in } \Omega, \\ u=g_1 & \text{on } \Gamma_d, \\ (a\nabla u)\cdot\boldsymbol{n}=g_2 & \text{on } \Gamma_n, \end{cases} \tag{1.1}$$

where Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d (d=2,3)$ with Lipschitz continuous boundary $\partial\Omega$, Γ_d and Γ_n are polygonal subsets of the boundary $\partial\Omega$, $\boldsymbol{n}$ is a unit outward normal direction to $\partial\Omega$, $f\in L^2(\Omega)$. The coefficient $a(\boldsymbol{x})\in W^{1,\infty}(\Omega)$ is assumed to be bounded in Ω, i.e., there exist positive constants $a_{\min}$ and $a_{\max}$ such that $a_{\min}\le a(\boldsymbol{x})\le a_{\max}, x\in\Omega$. The boundary data g_1 and g_2 are two given functions defined on Γ_d and Γ_n,

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respectively. We denote the complement of the Neumann boundary by $\Gamma_n^c := \partial\Omega \backslash \Gamma_n$. We assume that problem (1.1) is ill-posed, that is, $\Gamma_d \cap \Gamma_n \neq \emptyset$ or $\Gamma_d \cup \Gamma_n \neq \partial\Omega$.

Contrary to a well-posed elliptic problem, the ill-posedness of the problem (1.1) results from some special practical applications, where the Dirichlet data g_1 and the Neumann data g_2 are both available on a common part of domain boundary (i.e., $\Gamma_d \cap \Gamma_n \neq \emptyset$), and the boundary conditions or its data may be lost on a part of domain boundary (i.e., $\Gamma_d \cup \Gamma_n \neq \partial\Omega$). For instance, the elliptic Cauchy problem plays a crucial role to use the electrical impedance tomography for noninvasive detection [7, 23]. In this application, a weak current is applied to the electrodes on the surface of the human body and then the voltage values on the electrodes is measured. It means that $\Gamma_d = \Gamma_n$ in the model problem. According to the relationship between the voltage on Γ_d and the current on Γ_n, the internal electrical impedance of the human body or the change value of electrical impedance can be reconstructed. For more applications and relative results on elliptic Cauchy problem we refer to [2, 9–11, 19, 20, 22] and the references cited therein.

It is well known that the Cauchy problem defined as (1.1) is severely ill-posed [6] and even when a solution exists, it does not depend continuously on the boundary data. Therefore, how to design accuracy computational methods for approximating the ill-posed elliptic Cauchy problem remains a challenging topic. Most numerical methods are designed based on the well-posedness of the physical model problem. Following this basic idea, an important strategy for numerically approximating the elliptic Cauchy problem is to regularize the ill-posed problem to obtain a well-posed problem. The reduced well-posed problem then can be solved numerically using standard approximation techniques, such as finite element methods (FEM), boundary element methods and hybrid methods. We refer to [3, 4, 8, 12, 13, 24] for regularization methods and related approximation on the ill-posed Cauchy problem.

In 2013, Burman introduced the primal-dual stabilized finite element methods for ill-posed elliptic problem in [5]. This primal-dual method discretized the ill-posed problem through a constrained optimization problem. The unstable discrete problem was then stabilized by using techniques known from the theory of the stabilized finite element methods. Later, the method was further developed for the approximation of elliptic data assimilation problems [14], parabolic data reconstruction problems [15, 16], well-posed convection-diffusion problems [18]. As an extension of this method, a primal-dual weak Galerkin finite element methods were proposed in [25, 26] by employing weak finite element functions to approximate the solution of the elliptic Cauchy problem.

Motivated by the work of Burman [5], we develop a new primal-dual method which is based on the discontinuous Galerkin finite element spaces. Although the idea in [5] can be applied to any finite dimensional space, we point out that our techniques completely differ from Burman [5] in the numerical analysis. Specifically, the convergence analysis in [5] is based on the conditional stability estimates for the exact solution of Cauchy problem, while our analysis is derived by the consistency of the discrete scheme (in the sense that the exact solution satisfies the discrete system) and a special norm on the discontinuous Galerkin finite element spaces. One of our main results indicates that it attains a unique solution for the discrete scheme if the solution of the ill-posed elliptic Cauchy problems is unique. Another main result is that our method enjoys the optimal convergence order in a discrete Sobolev norm.

Throughout this paper, for any $m\geq 0$, we adopt the notation $H^m(D)$ to indicate the usual Sobolev space on domain $D\subseteq\Omega$ equipped with the norm $\|\cdot\|_{m,D}$ and seminorm $|\cdot|_{m,D}$. The inner product in $H^m(D)$ is denoted by $(\cdot,\cdot)_{m,D}$. The space $H^0(D)$ coincides with $L^2(D)$, whose norm and inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D=\Omega$,we omit the subscript D.

The rest of the paper is organized as follows. Section 2 introduces the model Cauchy problem together with the existence and uniqueness theory of the solution. In Section 3, the primal-dual discontinuous Galerkin finite element method is established for solving the ill-posed elliptic Cauchy problem; the existence and uniqueness of the numerical solution are proved. Section 4 is devoted to the error analysis for the discrete scheme. In Section 5, some numerical examples are provided to illustrate the performance of the proposed method.

2 The elliptic Cauchy problem

From the Schwartz reflection principle [21], we know that if the boundary data g_1 and g_2 are chosen arbitrarily, the existence of solution can not be guaranteed. However, there exists a dense subset M of $H^{\frac{1}{2}}(\Gamma_d)\times H^{-\frac{1}{2}}(\Gamma_n)$ such that the problem (1.1) has a solution for any Cauchy data $(g_1,g_2)\in M$. Moreover, if a solution exists, it must be unique under the condition that the $(d-1)$-dimensional measure of $\Gamma_d\cap\Gamma_n$ is non-zero. For the complete results, as well as full proofs, we refer to [3]. Here we only present the result on the uniqueness of the solution, which is useful for our numerical analysis.

Lemma 2.1. *Assume that Ω is an open bounded and connected domain in $\mathbb{R}^d (d=2,3)$ with Lipschitz continuous boundary $\partial\Omega$. Denote by Γ_d the portion of the Dirichlet boundary and Γ_n the Neumann portion. Assume that $\Gamma_d\cap\Gamma_n$ is a non-trivial portion of $\partial\Omega$. Then, the solutions of the elliptic Cauchy problem* (1.1)*, if they exist, are unique.*

Throughout this paper, we assume that data Cauchy data $g_1\times g_2\in M$ and $\Gamma_d\cap\Gamma_n$ is a nontrivial portion of the domain boundary such that there exists a unique solution $u\in H^{3/2}(\Omega)$ for the elliptic Cauchy problem (1.1).

3 Primal-dual discontinuous Galerkin finite element formulation

Let $\mathcal{T}_h=\cup\{K\}$ be a regular triangular or tetrahedral mesh of Ω parameterized by mesh size $h=\max_{K\in\mathcal{T}_h}\{h_K\}$, where h_K is the diameter of element K. Denote by $\mathcal{E}_h=\cup\{e:e\subset\partial K,\ K\in\mathcal{T}_h\}$ the union of all edges/faces of elements in $\mathcal{T}_h$ and set $\mathcal{E}_h^0=\mathcal{E}_h\setminus\partial\Omega$. For any edge/face $e\in\mathcal{E}_h$, denoted by h_e its length.

To define the discontinuous Galerkin finite element scheme, we need to introduce the average and jump on the element boundaries. Let K_1 and K_2 be two adjacent elements with common edge/face e, and let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the unit normal vectors on e exterior to K_1 and K_2, respectively. For a scalar q, we define the average $\{\cdot\}$ and the jump $[\cdot]$ on $e\in\mathcal{E}_h$ by

$$\begin{aligned} &\{q\}=\frac{1}{2}(q|_{K_1}+q|_{K_2}), \qquad e\in\mathcal{E}_h^0, \qquad \{q\}=q, \qquad e\subset\partial\Omega, \\ &[q]=q|_{K_1}\boldsymbol{n}_1+q|_{K_2}\boldsymbol{n}_2, \qquad e\in\mathcal{E}_h^0, \qquad [q]=q\boldsymbol{n}, \qquad e\subset\partial\Omega. \end{aligned}$$

Similarly, for a vector function $\boldsymbol{w}$, we define

$$\begin{aligned} &\{\boldsymbol{w}\}=\frac{1}{2}(\boldsymbol{w}|_{K_1}+\boldsymbol{w}|_{K_2}), \qquad e\in\mathcal{E}_h^0, \qquad \{\boldsymbol{w}\}=\boldsymbol{w}, \qquad e\subset\partial\Omega, \\ &[\boldsymbol{w}]=\boldsymbol{w}|_{K_1}\cdot\boldsymbol{n}_1+\boldsymbol{w}|_{K_2}\cdot\boldsymbol{n}_2, \qquad e\in\mathcal{E}_h^0, \qquad [\boldsymbol{w}]=\boldsymbol{w}\cdot\boldsymbol{n}, \qquad e\subset\partial\Omega. \end{aligned}$$

For a given division $\mathcal{T}_h$, we define the discontinuous finite element spaces:

$$V_h=\{v\in L^2(\Omega): v|_K\in P_r(K),\ \forall K\in\mathcal{T}_h\},$$

where $P_r(K)$ denotes the set of all polynomials of degree no more than r defined on K.

To simplify the notations, we use the following L^2 inner products and norms:

$$\begin{aligned} &(v,w)_h=\sum_{K\in\mathcal{T}_h}(v,w)_K, \qquad \|v\|_h^2=(v,v)_h, \\ &\langle v,w\rangle_e=\int_e vw\mathrm{d}s, \qquad \|v\|_e^2=\langle v,v\rangle_e. \end{aligned}$$

For a subset $\mathcal{S}\subset\mathcal{E}_h$, we also use

$$\langle w,v\rangle_{\mathcal{S}}=\sum_{e\in\mathcal{S}}\langle w,v\rangle_e.$$

Multiplying the first equation of the system (1.1) by $v_h\in V_h$ and using integration by parts, we obtain

$$\sum_{K\in\mathcal{T}_h}(a\nabla u,\nabla v_h)_K-\sum_{K\in\mathcal{T}_h}\langle(a\nabla u)\cdot\boldsymbol{n},v_h\rangle_{\partial K}=(f,v_h)_h. \tag{3.1}$$

For all $v\in\prod_{K\in\mathcal{T}_h}L^2(\partial K)$ and all $\boldsymbol{q}\in\prod_{K\in\mathcal{T}_h}L^2(\partial K)^d$, a straightforward computation gives (cf. [27])

$$\sum_{K\in\mathcal{T}_h}\langle\boldsymbol{q}\cdot\boldsymbol{n},v\rangle_{\partial K}=\sum_{e\in\mathcal{E}_h}\langle\{\boldsymbol{q}\},[v]\rangle_e+\sum_{e\in\mathcal{E}_h^0}\langle[\boldsymbol{q}],\{v\}\rangle_e. \tag{3.2}$$

Applying (3.2) and noting the fact that $[a\nabla u]=0$ for $u\in H^{3/2}(\Omega)$ on $e\in\mathcal{E}_h^0$, we have from (3.1) that

$$(a\nabla u,\nabla v_h)_h-\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}\langle\{a\nabla u\},[v_h]\rangle_e=(f,v_h)_h+\sum_{e\in\Gamma_n}\langle g_2,v_h\rangle_e. \tag{3.3}$$

Thus we derive a discrete scheme: find $u_h \in V_h$ with $u_h = g_1^h$ on Γ_d such that

$$a_h(u_h,v_h) = (f,v_h)_h + \sum_{e\in\Gamma_n} \langle g_2,v_h\rangle_e, \quad \forall v_h \in V_h, \tag{3.4}$$

where

$$a_h(w,v) = (a\nabla w,\nabla v)_h - \sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c} \langle\{a\nabla w\},[v]\rangle_e,$$

and g_1^h is the discontinuous finite element approximation of g_1.

Unfortunately, the discrete problem (3.4) is not well-posed, even if we follow the technique of the discontinuous finite element method to add an appropriate stabilizer. To overcome this difficulty, we apply the strategy in [5] to present a primal-dual discontinuous finite element method. First, we couple the discrete problem (3.4) with its dual problem which seeks $\lambda_h \in V_h$ such that

$$a_h(v_h,\lambda_h) = 0, \quad \forall v_h \in V_h, \tag{3.5}$$

where λ_h is the Lagrange multiplier. Next, we stabilize the primal-dual equations (3.4)-(3.5) via some appropriate stabilizers which should include the stabilized terms for discontinuous finite element, boundary conditions, as well as boundary data.

Let $V(h)=V_h+H^{3/2}(\Omega)$. For any $u,v\in V(h)$, we introduce the following bilinear forms

$$\begin{aligned}
s_d(u,v) &= \sum_{e\in\mathcal{E}_h^0\cup\Gamma_d} h_e^{-1}\langle[u],[v]\rangle_e,\\
s_n(u,v) &= \sum_{e\in\mathcal{E}_h^0\cup\Gamma_n} h_e\langle[\nabla u],[\nabla v]\rangle_e,\\
s(u,v) &= \sum_{e\in\mathcal{E}_h^0\cup\Gamma_d\cup\Gamma_n^c} h_e^{-1}\langle[u],[v]\rangle_e.
\end{aligned}$$

Now, we define the primal-dual discontinuous Galerkin finite element approximation for problem (1.1): find $(u_h,\lambda_h)\in V_h\times V_h$ such that

$$s_d(u_h,v)+s_n(u_h,v)-a_h(v,\lambda_h) = \sum_{e\in\Gamma_d} h_e^{-1}\langle g_1,v\rangle_e + \sum_{e\in\Gamma_n} h_e\left\langle a^{-1}g_2,\frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_e, \quad \forall v\in V_h, \tag{3.6a}$$

$$s(\lambda_h,w)+a_h(u_h,w) = (f,w)_h + \sum_{e\in\Gamma_n}\langle g_2,w\rangle_e, \quad \forall w\in V_h. \tag{3.6b}$$

It is easy to check that the solution $u\in H^{3/2}(\Omega)$ of problem (1.1) together with $\lambda=0$ satisfies

$$s_d(u,v)+s_n(u,v)-a_h(v,\lambda) = \sum_{e\in\Gamma_d} h_e^{-1}\langle g_1,v\rangle_e + \sum_{e\in\Gamma_n} h_e\left\langle a^{-1}g_2,\frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_e, \quad \forall v\in V_h, \tag{3.7a}$$

$$s(\lambda,w)+a_h(u,w) = (f,w)_h + \sum_{e\in\Gamma_n}\langle g_2,w\rangle_e, \quad \forall w\in V_h. \tag{3.7b}$$

The following result states the well-posedness of the discrete problem (3.6a)-(3.6b).

Theorem 3.1. *Assume that the solution of the elliptic Cauchy problem* (1.1) *exists and* $\Gamma_d \cap \Gamma_n$ *is a non-trivial portion of* $\partial\Omega$. *Then the primal-dual discontinuous Galerkin finite element scheme* (3.6a)-(3.6b) *has a unique solution pair* $(u_h,\lambda_h) \in V_h \times V_h$.

Proof. Since the discrete problem (3.6a)-(3.6b) is essentially a linear system in which the number of the equations is the same as the number of unknowns, it suffices to prove that the homogeneous problem has only the trivial solution. Let $f=0$, $g_1=0$ and $g_2=0$ in (3.6a)-(3.6b). Taking $v=u_h$ in (3.6a), $w=\lambda_h$ in (3.6b) and adding the two equations, we obtain

$$s_d(u_h,u_h)+s_n(u_h,u_h)+s(\lambda_h,\lambda_h)=0.$$

By the definitions of bilinear forms $s_d(\cdot,\cdot)$, $s_n(\cdot,\cdot)$ and $s(\cdot,\cdot)$, we get

$$s_d(u_h,u_h)=s_n(u_h,u_h)=s(\lambda_h,\lambda_h)=0,$$

which further reduces to

$$u_h=0, \quad e\subset\Gamma_d, \quad \frac{\partial u_h}{\partial \boldsymbol{n}}=0, \quad e\subset\Gamma_n, \quad [u_h]_e=[\nabla u_h]_e=0, \quad e\in\mathcal{E}_h^0, \tag{3.8a}$$

$$\lambda_h=0, \quad e\subset\Gamma_d\cup\Gamma_n^c, \quad [\lambda_h]_e=0, \quad e\in\mathcal{E}_h^0. \tag{3.8b}$$

It follows from (3.8a), (3.8b), (3.6a) and (3.6b) that

$$\begin{aligned} &a_h(v,\lambda_h)=0, \qquad \forall v\in V_h,\\ &a_h(u_h,w)=0, \qquad \forall w\in V_h, \end{aligned}$$

which together with (3.8b) leads to

$$(a\nabla v,\nabla\lambda_h)_h=0, \qquad \forall v\in V_h, \tag{3.9a}$$

$$(a\nabla u_h,\nabla w)_h-\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}\langle\{a\nabla u_h\},[w]\rangle_e=0, \qquad \forall w\in V_h. \tag{3.9b}$$

Letting $v=\lambda_h$ in (3.9a), we have

$$(a\nabla\lambda_h,\nabla\lambda_h)_h=0,$$

which implies λ_h is piecewise constant on $\mathcal{T}_h$. Hence we have $\lambda_h\equiv 0$ in Ω from (3.8b).

It remains to show that $u_h\equiv 0$ in Ω. First, we consider the case $u_h|_K\in P_1(K), K\in\mathcal{T}_h$. From (3.8a), it is easy to verify that $u_h\equiv 0$ in Ω. The next is to consider the case $u_h|_K\in P_r(K)$, $K\in\mathcal{T}_h$, $r>1$. It follows from (3.8a) that $u_h\in C^1(\Omega)\cap H^2(\Omega)$. Using (3.9b), (3.2) and integration by parts, we have

$$-(\nabla\cdot(a\nabla u_h),w)=0, \quad \forall w\in V_h. \tag{3.10}$$

Let $a(\boldsymbol{x})$ be a polynomial of degree no more than two. Taking $w=\nabla\cdot(a\nabla u_h)$ in (3.10), we obtain

$$-\nabla\cdot(a\nabla u_h)=0 \quad \text{in } \Omega. \tag{3.11}$$

From (3.8a) and (3.11), u_h satisfies the following elliptic Cauchy problem

$$\begin{aligned} &-\nabla\cdot(a\nabla u_h)=0 && \text{in } \Omega,\\ &u_h=0 && \text{on } \Gamma_d,\\ &(a\nabla u_h)\cdot\boldsymbol{n}=0 && \text{on } \Gamma_n. \end{aligned}$$

It follows from Lemma 2.1 that $u_h\equiv 0$ in Ω when $\Gamma_d\cap\Gamma_n\neq\emptyset$. We complete the proof of the theorem. □

4 Error estimate

This section concerns the error analysis for the primal-dual discontinuous Galerkin finite element method defined in the previous section.

Introduce the following semi-norms for $v\in V(h)$:

$$|v|_{\Gamma_d}^2=s_d(v,v),\quad |v|_{\Gamma_n}^2=s_n(v,v),\quad |v|_{\Gamma}^2=s(v,v),\quad |u|_{1,h}^2=(\nabla u,\nabla u)_h.$$

The following trace inequality can be found in [1]. For $w\in H^1(K)$ and for an edge e of K,

$$\|w\|_e^2\le C_{tra}(h_e^{-1}|w|_K^2+h_e|w|_{1,K}^2),\quad K\in\mathcal{T}_h. \tag{4.1}$$

Let $P_h:\phi\in L^2(\Omega)\to P_h\phi\in V_h$ be the L^2 projection defined by

$$(\phi-P_h\phi,q)_K=0,\quad \forall q\in P_r(K),\quad K\in\mathcal{T}_h.$$

It can be verified that the operator enjoys the approximating property [28]:

$$\|\phi-P_h\phi\|_K+h_K\|\nabla(\phi-P_h\phi)\|_K\le Ch_K^m\|\phi\|_{m,K},\quad 1\le m\le r+1. \tag{4.2}$$

The following lemma is important in the error analysis.

Lemma 4.1. *Let $(u,\lambda=0)$ be the solution of the variational problem* (3.7a)-(3.7b) *and (u_h,λ_h) be the solution of primal-dual discontinuous Galerkin finite element method* (3.6a)-(3.6b). *Let $e_u=u_h-P_hu$ and $e_\lambda=\lambda_h-\lambda=\lambda_h$. We assume $u\in H^{r+1}(\Omega)$, $r\ge 1$. Then*

$$|e_\lambda|_{1,h}^2\le C\big(|e_u|_{\Gamma_d}^2+|e_u|_{\Gamma_n}^2+|e_\lambda|_{\Gamma}^2+h^{2r}\|u\|_{r+1}^2\big), \tag{4.3}$$

where C depends on C_{tra}, $a_{\min}$ and $a_{\max}$.

Proof. Subtracting (3.7a) from (3.6a), we obtain the following error equation:

$$s_d(u_h-u,v)+s_n(u_h-u,v)-a_h(v,\lambda_h)=0,\quad \forall v\in V_h, \tag{4.4}$$

which can be rewritten as

$$s_d(e_u,v)+s_n(e_u,v)-a_h(v,e_\lambda)=s_d(u-P_hu,v)+s_n(u-P_hu,v),\quad \forall v\in V_h. \tag{4.5}$$

Setting $v=e_\lambda$ in (4.5) and using the definition of the bilinear form $a_h(\cdot,\cdot)$, we get

$$\begin{aligned} a_{\min}|e_\lambda|_{1,h}^2 \le & s_d(e_u,e_\lambda)+s_n(e_u,e_\lambda)-s_d(u-P_hu,e_\lambda) \\ & -s_n(u-P_hu,e_\lambda)+\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}\langle\{a\nabla e_\lambda\},[e_\lambda]\rangle_e. \end{aligned} \tag{4.6}$$

It follows from (4.6), Young's inequality, trace inequality (4.1) and the inverse inequality that

$$\begin{aligned} a_{\min}|e_\lambda|_{1,h}^2 \le & |e_u|_{\Gamma_d}|e_\lambda|_\Gamma+|e_u|_{\Gamma_n}|e_\lambda|_{\Gamma_n}+|u-P_hu|_{\Gamma_d}|e_\lambda|_\Gamma+|u-P_hu|_{\Gamma_n}|e_\lambda|_{\Gamma_n} \\ & +\Big(\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla e_\lambda\}\|_e^2\Big)^{\frac{1}{2}}|e_\lambda|_\Gamma \\ \le & \frac{1}{2}|e_u|_{\Gamma_d}^2+\frac{1}{2}|e_\lambda|_\Gamma^2+\frac{1}{2\varepsilon}|e_u|_{\Gamma_n}^2+\frac{\varepsilon}{2}|e_\lambda|_{\Gamma_n}^2+\frac{1}{2}|u-P_hu|_{\Gamma_d}^2+\frac{1}{2}|e_\lambda|_\Gamma^2 \\ & +\frac{1}{2\varepsilon}|u-P_hu|_{\Gamma_n}^2+\frac{\varepsilon}{2}|e_\lambda|_{\Gamma_n}^2+\varepsilon\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla e_\lambda\}\|_e^2+\frac{1}{4\varepsilon}|e_\lambda|_\Gamma^2 \\ \le & \frac{1}{2}|e_u|_{\Gamma_d}^2+\frac{1}{2\varepsilon}|e_u|_{\Gamma_n}^2+\Big(1+\frac{1}{4\varepsilon}\Big)|e_\lambda|_\Gamma^2+\frac{1}{2}|u-P_hu|_{\Gamma_d}^2 \\ & +\frac{1}{2\varepsilon}|u-P_hu|_{\Gamma_n}^2+\varepsilon C_{tra}\max(1,a_{\max}^2)\|\nabla e_\lambda\|^2, \end{aligned} \tag{4.7}$$

and choosing ε small enough, we have

$$|e_\lambda|_{1,h}^2\le C\big(|e_u|_{\Gamma_d}^2+|e_u|_{\Gamma_n}^2+|e_\lambda|_\Gamma^2+|u-P_hu|_{\Gamma_d}^2+|u-P_hu|_{\Gamma_n}^2\big),$$

where C depends on C_{tra}, $a_{\min}$ and $a_{\max}$. The proof is completed by using the trace inequality (4.1) and the approximating property (4.2). □

Theorem 4.1. *Under the assumptions of Lemma* 4.1*, there exists a constant* $C>0$ *such that the following error estimates hold*

$$|u_h-P_hu|_{\Gamma_d}+|u_h-P_hu|_{\Gamma_n}\le Ch^r\|u\|_{r+1}, \tag{4.8a}$$
$$|\lambda_h|_\Gamma+|\lambda_h|_{1,h}\le Ch^r\|u\|_{r+1}. \tag{4.8b}$$

Proof. Subtracting (3.7b) from (3.6b), we obtain

$$s(\lambda_h-\lambda,w)+a_h(u_h-u,w)=0,\quad \forall w\in V_h, \tag{4.9}$$

which implies

$$s(e_\lambda,w)+a_h(e_u,w)=a_h(u-P_hu,w),\quad \forall w\in V_h. \tag{4.10}$$

Taking $v=e_u$ in (4.5), $w=e_\lambda$ in (4.10), and adding the two equations lead to

$$|e_u|_{\Gamma_d}^2+|e_u|_{\Gamma_n}^2+|e_\lambda|_\Gamma^2=s_d(u-P_hu,e_u)+s_n(u-P_hu,e_u)+a_h(u-P_hu,e_\lambda).$$

Then, we have from the Cauchy-Schwarz and Young's inequality that

$$
\begin{aligned}
&|e_u|^2_{\Gamma_d}+|e_u|^2_{\Gamma_n}+|e_\lambda|^2_\Gamma\\
\le&|u-P_hu|_{\Gamma_d}|e_u|_{\Gamma_d}+|u-P_hu|_{\Gamma_n}|e_u|_{\Gamma_n}+a_{\max}|u-P_hu|_{1,h}|e_\lambda|_{1,h}\\
&+\Big(\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla(u-P_hu)\}\|_e^2\Big)^{\frac{1}{2}}|e_\lambda|_\Gamma\\
\le&\frac{1}{2}|e_u|^2_{\Gamma_d}+\frac{1}{2}|u-P_hu|^2_{\Gamma_d}+\frac{1}{2}|e_u|^2_{\Gamma_n}+\frac{1}{2}|u-P_hu|^2_{\Gamma_n}\\
&+\frac{a_{\max}}{2\alpha}|u-P_hu|^2_{1,h}+\frac{a_{\max}\alpha}{2}|e_\lambda|^2_{1,h}\\
&+\frac{1}{2}\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla(u-P_hu)\}\|_e^2+\frac{1}{2}|e_\lambda|^2_\Gamma,
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
&|e_u|^2_{\Gamma_d}+|e_u|^2_{\Gamma_n}+|e_\lambda|^2_\Gamma\\
\le&|u-P_hu|^2_{\Gamma_d}+|u-P_hu|^2_{\Gamma_n}+\frac{a_{\max}}{\alpha}|u-P_hu|^2_{1,h}+a_{\max}\alpha|e_\lambda|^2_{1,h}\\
&+\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla(u-P_hu)\}\|_e^2,
\end{aligned}
\tag{4.11}
$$

where $\alpha>0$ is a parameter.

For sufficiently small α, using (4.11) and Lemma 4.1, we have

$$
\begin{aligned}
&|e_u|^2_{\Gamma_d}+|e_u|^2_{\Gamma_n}+|e_\lambda|^2_\Gamma\\
\le&C\Big(|u-P_hu|^2_{\Gamma_d}+|u-P_hu|^2_{\Gamma_n}+|u-P_hu\|^2_{1,h}\\
&\quad+\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla(u-P_hu)\}\|_e^2+h^{2r}\|u\|^2_{r+1}\Big).
\end{aligned}
\tag{4.12}
$$

It follows from (4.12), the trace inequality (4.1) and the approximating property (4.2) that

$$
|e_u|^2_{\Gamma_d}+|e_u|^2_{\Gamma_n}+|e_\lambda|^2_\Gamma\le Ch^{2r}\|u\|^2_{r+1},
\tag{4.13}
$$

which gives (4.8a).

Substituting (4.13) into (4.3), we obtain

$$
|e_\lambda|^2_{1,h}\le Ch^{2r}\|u\|^2_{r+1}.
\tag{4.14}
$$

The proof of (4.8b) is completed by combining (4.13) and (4.14). □

We define a norm $|||\cdot|||$ for $v\in V_h$ as follows:

$$
|||v|||^2=|v|^2_{\Gamma_d}+|v|^2_{\Gamma_n}+\sup_{w\in V_h}\frac{|a_h(v,w)|}{\|w\|_{1,\Gamma}},
$$

where $\|v\|_{1,\Gamma}^2=|v|_{1,h}^2+|v|_\Gamma^2$. Tracking the proof of Theorem 3.1, it is easy to check that $|||\cdot|||$ defines a norm in the discontinuous finite element spaces V_h.

Now, we can derive the error estimate for the primal-dual discontinuous Galerkin finite solution in the following theorem.

Theorem 4.2. *Let u and u_h be the solutions of* (1.1) *and* (3.6a)-(3.6b)*, respectively. There exists a constant C such that the following error estimate holds:*

$$|||u_h-P_hu|||\le Ch^r\|u\|_{r+1}.$$

Proof. It follows from (4.10), Cauchy-Schwarz inequality, the trace inequality (4.1), the approximating property (4.2) and the estimate (4.8b) that

$$\begin{aligned}|a_h(e_u,w)|\le&|s(e_\lambda,w)|+|a_h(u-P_hu,w)|\\ \le&|e_\lambda|_\Gamma|w|_\Gamma+a_{\max}|u-P_hu|_{1,h}|w|_{1,h}\\ &+\Big(\sum_{e\in\mathcal{E}_h^0\cup\Gamma_n^c}h_e\|\{a\nabla(u-P_hu)\}\|_e^2\Big)^{\frac{1}{2}}|w|_\Gamma\\ \le&Ch^r\|u\|_{r+1}\|w\|_{1,\Gamma},\end{aligned}$$

which implies

$$\sup_{w\in V_h}\frac{|a_h(e_u,w)|}{\|w\|_{1,\Gamma}}\le Ch^r\|u\|_{r+1}. \tag{4.15}$$

Using estimates (4.8a) and (4.15), we have

$$|||e_u|||\le Ch^r\|u\|_{r+1}, \tag{4.16}$$

which completes the proof. □

Corollary 4.1. *Let u and u_h be the solutions of* (1.1) *and* (3.6a)-(3.6b)*, respectively. There exists a constant C such that the following error estimate holds:*

$$|||u-u_h|||\le Ch^r\|u\|_{r+1}.$$

Proof. By the definition of the triple norm, Cauchy-Schwarz inequality, the trace inequality (4.1), the inverse inequality and the approximating property (4.2), we obtain

$$|||u-P_hu|||\le Ch^r\|u\|_{r+1},$$

which completes the proof by applying Theorem 4.2 and the triangle inequality. □

5 Numerical experiments

In this section, we provide some numerical results to show the performance of the proposed method.

Let us consider problem (1.1) on the domain $\Omega=(0,1)^2$ with the diffusion coefficient $a(\boldsymbol{x})=1$. The source term $f=-\nabla\cdot(a\nabla u)$ in Ω and the Cauchy boundary data $g_1=u$ on Γ_d and $g_2=a\nabla u\cdot\boldsymbol{n}$ on Γ_n are chosen by the given exact solution $u=u(\boldsymbol{x})$.

In the numerical experiments, we first partition Ω into $N\times N$ uniform squares, and then each square is divided into two triangles by its diagonal line, resulting in a mesh $\mathcal{T}_h$ with size $\sqrt{2}/N$. The refined mesh $\mathcal{T}_{h/2}$ is obtained by connecting the midpoint of each edge of elements in $\mathcal{T}_h$ with straight line, and this procedure generates a mesh series $\mathcal{T}_{h/2^j}$, $j=0,1,\cdots$. The piecewise linear polynomial spaces are used in the discontinuous Galerkin finite element method (3.6a)-(3.6b) solving the elliptic Cauchy problem (1.1). The error between the numerical solution of (3.6a)-(3.6b) and the L^2 projection of the exact solution for elliptic Cauchy problem (1.1) is computed in the following norms:

$$\begin{aligned}
\|u_h-P_hu\|_h&=\Big(\sum_{K\in\mathcal{T}_h}(u_h-P_hu,u_h-P_hu)_K\Big)^{\frac{1}{2}},\\
\|u_h-P_hu\|_{1,h}&=\Big(\|u_h-P_hu\|_h^2+\|\nabla(u_h-P_hu)\|_h^2\Big)^{\frac{1}{2}},\\
\|u_h-P_hu\|_{1,\Gamma}&=\Big(\|\nabla(u_h-P_hu)\|_h^2+|u_h-P_hu|_\Gamma^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Tables 1-4 show the numerical results for different exact solutions when the boundary condition is set by

$$\Gamma_d:=\{x=0;\ y\in(0,1)\}\cup\{x=1;\ y\in(0,1)\}\cup\{x\in(0,1);\ y=0\}$$

and

$$\Gamma_n:=\{x\in(0,1);\ y=0\}\cup\{x\in(0,1);\ y=1\}\cup\{x=1;\ y\in(0,1)\}$$

as shown in Fig. 1. We see that the primal-dual discontinuous Galerkin finite element method has optimal $\mathcal{O}(h^2)$ accuracy with respect to $\|\cdot\|_h$, and $\mathcal{O}(h)$ accuracy with respect to $\|\cdot\|_{1,h}$ and $\|\cdot\|_{1,\Gamma}$.

In Table 5, we list some numerical results when the exact solutions are given by

$$u_1(x,y)=x^2+y^2-10xy,\quad u_2(x,y)=\sin(x)\sin(y)\quad \text{and}\quad u_3(x,y)=\cos(x)\cos(y),$$

respectively. In this experiment, we chose

$$\Gamma_d:=\{x=0;\ y\in(0,1)\}\cup\{x=1;\ y\in(0,1)\}\cup\{x\in(0,1),\ y=0\},$$

and

$$\Gamma_n = \Gamma_d,$$

see Fig. 2. It can be easily observed that the convergence rate in $\|\cdot\|_{1,h}$ is of the optimal

Table 1: Convergence rates for the exact solution $u=30xy(1-x)(1-y)$.

N	$\|u_h-P_hu\|_h$		$\|u_h-P_hu\|_{1,h}$		$\|u_h-P_hu\|_{1,\Gamma}$	
	error	Order	Error	Order	Error	Order
4	0.0542	N/A	1.3202	N/A	1.6000	N/A
8	0.0142	1.9297	0.6807	0.9556	0.8729	0.8741
16	0.0037	1.9611	0.3432	0.9882	0.4543	0.9424
32	9.3459e-04	1.9657	0.1720	0.9968	0.2315	0.9726
64	2.3891e-04	1.9679	0.0860	0.9989	0.1168	0.9866
128	6.0827e-05	1.9737	0.0430	0.9995	0.0587	0.9934

Table 2: Convergence rates for the exact solution $u=x^2+y^2-10xy$.

N	$\|u_h-P_hu\|_h$		$\|u_h-P_hu\|_{1,h}$		$\|u_h-P_hu\|_{1,\Gamma}$	
	error	Order	Error	Order	Error	Order
4	0.0508	N/A	1.0744	N/A	1.4922	N/A
8	0.0130	1.9681	0.5410	0.9900	0.7739	0.8741
16	0.0033	1.9721	0.2711	0.9968	0.3938	0.9424
32	8.4175e-04	1.9756	0.1356	0.9993	0.1986	0.9726
64	2.1316e-04	1.9814	0.0678	0.9999	0.0997	0.9866
128	5.3760e-05	1.9874	0.0339	1.0000	0.0500	0.9934

Table 3: Convergence rates for the exact solution $u=\sin(x)\sin(y)$.

N	$\|u_h-P_hu\|_h$		$\|u_h-P_hu\|_{1,h}$		$\|u_h-P_hu\|_{1,\Gamma}$	
	error	Order	Error	Order	Error	Order
4	0.0039	N/A	0.0824	N/A	0.1131	N/A
8	9.9720e-04	1.9787	0.0413	0.9985	0.0584	0.9533
16	2.5436e-04	1.9710	0.0206	1.0022	0.0296	0.9799
32	6.5042e-05	1.9674	0.0103	1.0020	0.0149	0.9911
64	1.6560e-05	1.9736	0.0051	1.0010	0.0075	0.9957
128	4.1908e-06	1.9824	0.0026	1.0004	0.0037	0.9978

Table 4: Convergence rates for the exact solution $u=\cos(x)\cos(y)$.

N	$\|u_h-P_hu\|_h$		$\|u_h-P_hu\|_{1,h}$		$\|u_h-P_hu\|_{1,\Gamma}$	
	error	Order	Error	Order	Error	Order
4	0.0037	N/A	0.0757	N/A	0.0905	N/A
8	8.7954e-04	2.0544	0.0374	1.0197	0.0456	0.9878
16	2.1840e-04	2.0098	0.0186	1.0095	0.0229	0.9934
32	5.5619e-05	1.9733	0.0092	1.0046	0.0115	0.9963
64	1.4273e-05	1.9623	0.0046	1.0020	0.0058	0.9978
128	3.6486e-06	1.9678	0.0023	1.0008	0.0029	0.9987

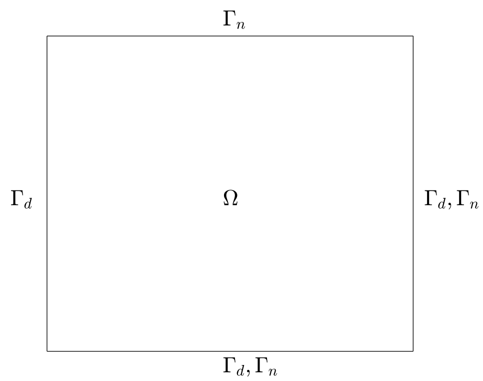


Figure 1: The computational domain and boundary conditions.

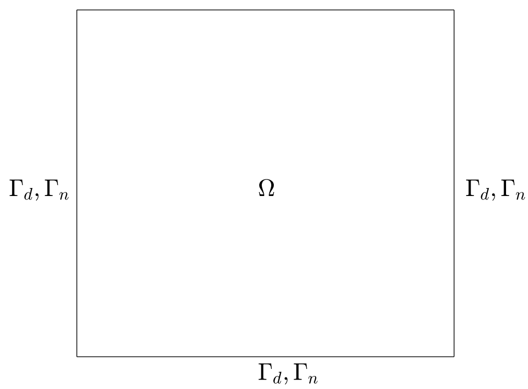


Figure 2: The computational domain and boundary conditions.

order $\mathcal{O}(h)$ for the different exact solution u_1, u_2 and u_3.

Table 6 shows the performance of the primal-dual discontinuous Galerkin method where the boundary conditions are chosen as B.C. I, B.C. II and B.C. III, see Fig. 3. Specifically,

- B.C. I: $\Gamma_d := \{x=0;\ y\in(0,1)\}\cup\{x\in(0,1);\ y=0\}$ and $\Gamma_n := \{x=1;\ y\in(0,1)\}\cup\{x\in(0,1);\ y=1\}$.
- B.C. II: $\Gamma_d := \{x\in(0,1);\ y=0\}\cup\{x\in(0,1);\ y=1\}\cup\{x=1;\ y\in(0,1)\}$ and $\Gamma_n=\Gamma_d$.
- B.C. III: $\Gamma_d := \{x=0;\ y\in(0,1)\}\cup\{x\in(0,1);\ y=0\}\cup\{x\in(0,1);\ y=1\}$ and $\Gamma_n := \{x\in(0,1);\ y=1\}\cup\{x=0;\ y\in(0,1)\}\cup\{x=1;\ y\in(0,1)\}$.

Note that the elliptic Cauchy problem (1.1) with the boundary condition, B.C. I, re-

Table 5: Convergence rates in norm $\|\cdot\|_{1,h}$ for different exact solutions.

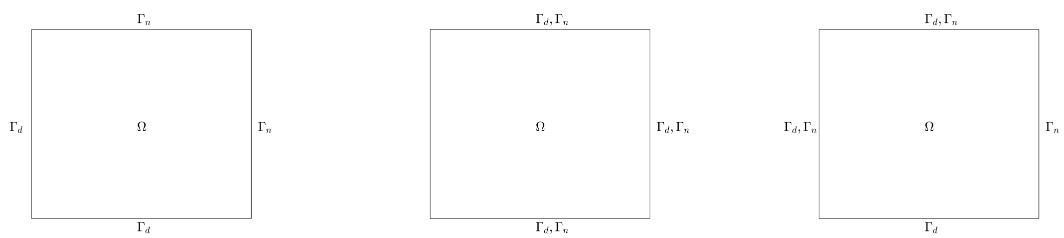
N	$\|u_h-P_hu_1\|_{1,h}$		$\|u_h-P_hu_2\|_{1,h}$		$\|u_h-P_hu_3\|_{1,h}$	
	error	Order	Error	Order	Error	Order
4	1.1025	N/A	0.0865	N/A	0.0815	N/A
8	0.5504	1.0022	0.0430	1.0094	0.0403	1.0168
16	0.2748	1.0020	0.0214	1.0041	0.0206	0.9682
32	0.1375	0.9990	0.0107	0.9983	0.0106	0.9522
64	0.0689	0.9961	0.0054	0.9862	0.0055	0.9523
128	0.0346	0.9925	0.0028	0.9669	0.0029	0.9461

Table 6: Convergence rates in norm $\|\cdot\|_{1,h}$ for the exact solution $u=x^2+y^2-10xy$ with different boundary conditions.

N	B.C. I		B.C. II		B.C. III	
	error	Order	Error	Order	Error	Order
4	1.0845	N/A	1.1025	N/A	1.0744	N/A
8	0.5438	0.9958	0.5504	1.0022	0.5410	0.9900
16	0.2719	1.0003	0.2748	1.0020	0.2711	0.9968
32	0.1358	1.0012	0.1375	0.9990	0.1356	0.9993
64	0.0679	1.0009	0.0689	0.9961	0.0678	0.9999
128	0.0339	1.0006	0.0347	0.9925	0.0339	1.0000

duces into a standard mixed boundary value problem, while the Cauchy problem with the other three boundary conditions is still ill-posed. The purpose of the experiment is to compare the efficiency of the primal-dual discontinuous Galerkin finite element methods for classical well-posed elliptic problem and the ill-posed elliptic Cauchy problem. The numerical results show that the convergence rate in $\|\cdot\|_{1,h}$ is of optimal order $\mathcal{O}(h)$ both for well-posed and ill-posed elliptic problem.

In practice, the boundary data always contains uncertainties, and thus it is essential to test the primal-dual discontinuous Galerkin finite element method with the noise data. In this test, the exact solution is chosen as $u=\cos(x)\cos(y)$ and the Dirichlet and Neu-



(a) B.C. I (b) B.C. II (c) B.C. III

Figure 3: The computational domain and boundary conditions.

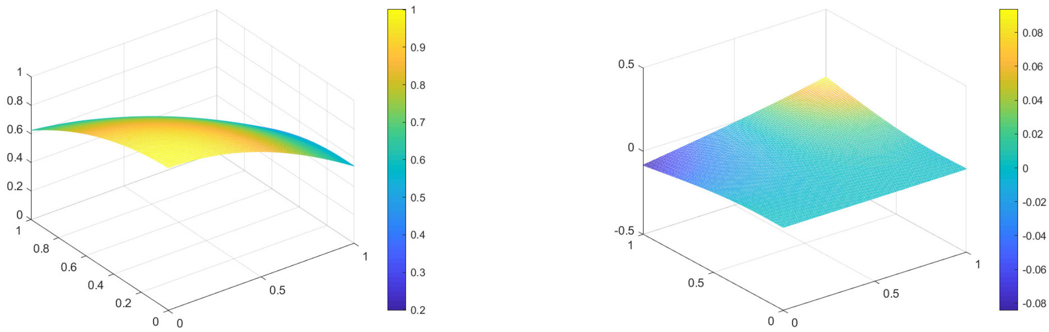


Figure 4: Numerical solution u_h (left) and error function $u-u_h$ (right) with the exact cauchy data.

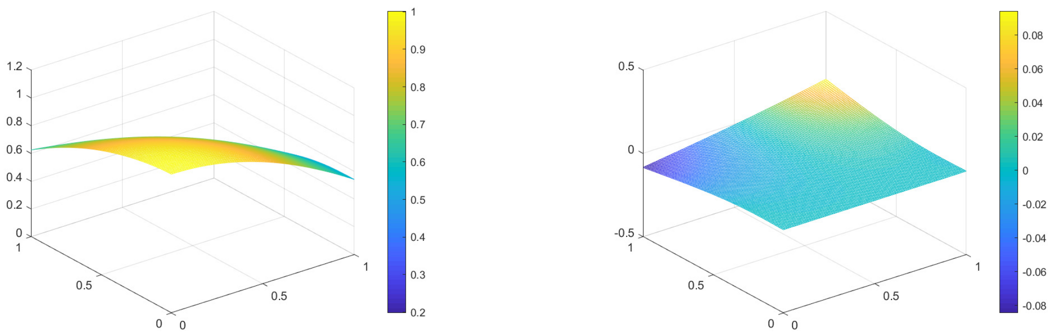


Figure 5: Numerical solution u_h (left) and error function $u-u_h$ (right) with a random perturbation by $0.001*(0.5-\text{rand})$ of the cauchy data.

mann boundary conditions are both imposed on $\{x\in(0,1),\ y=0\}$. Fig. 4 presents the numerical solution u_h and the error function $u-u_h$ with the exact boundary data computed according to the given solution $u=\cos(x)\cos(y)$. Figs. 5-7 show the numerical results when the boundary data is given by imposing a random noise $\nu*(0.5-Rand)$ on the exact boundary data. Here *Rand* is the MatLab function that generates rand numbers in the range $(0,1)$, and ν is a constant chosen as 0.001,0.01 and 0.1, respectively. Obviously, the increase in the amplitude of the random noise data has made the deviation of the error function from 0 become larger, but the proposed primal-dual discontinuous Galerkin finite element method work relatively well.

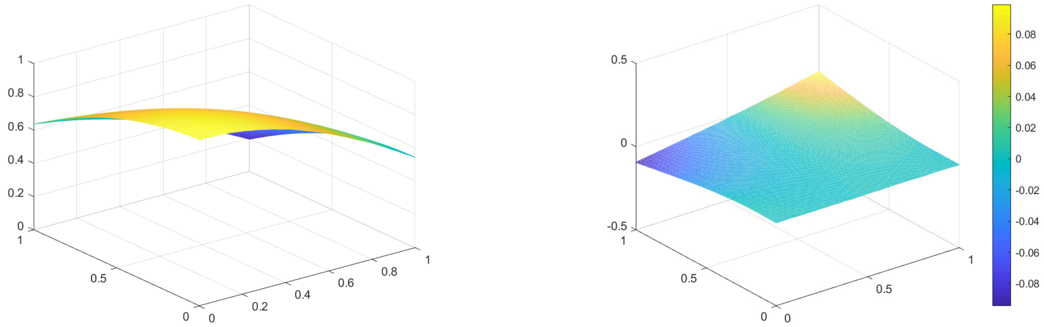


Figure 6: Numerical solution u_h (left) and error function $u-u_h$ (right) with a random perturbation by $0.01*(0.5-\text{rand})$ of the cauchy data.

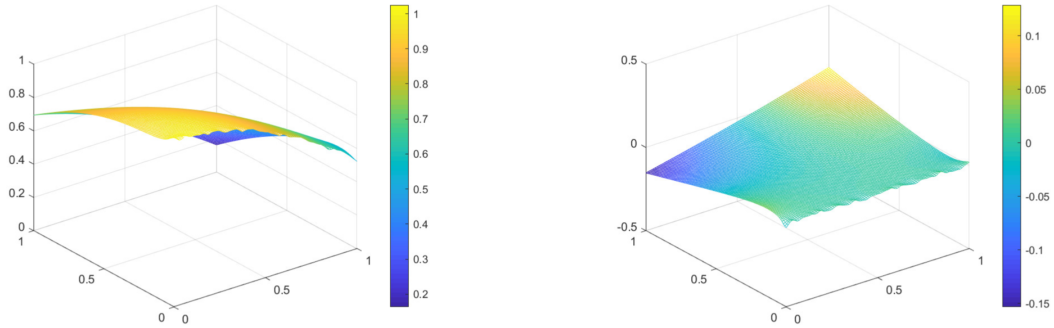


Figure 7: Numerical solution u_h (left) and error function $u-u_h$ (right) with a random perturbation by $0.1*(0.5-\text{rand})$ of the cauchy data.

6 Conclusions

In this paper, we have presented a primal-dual discontinuous Galerkin finite element method for a type of ill-posed elliptic Cauchy problem. It is shown that the primal-dual discontinuous Galerkin finite element method attains a unique solution, if the solution of the ill-posed elliptic Cauchy problems is unique. Based on the consistency of the discrete scheme (in the sense that the exact solution satisfies the discrete system), the optimal convergence in a discrete Sobolev norm is established. A possible future work is to extend our analysis to the adaptive primal-dual discontinuous Galerkin finite element method for solving the ill-posed elliptic Cauchy problem.

Acknowledgements

The work of Y. Chen is supported in part by China NSF grant No. 12001086. The author would like to thank the referees for the helpful suggestions.

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