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Nonpolynomial Jacobi Spectral-Collocation Method for Weakly Singular Fredholm Integral Equations of the Second Kind

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Abstract. In this paper a nonpolynomial Jacobi spectral-collocation (NJSC) method for the second kind Fredholm integral equations (FIEs) with weakly singular kernel $|s-t|^{-\gamma}$ is proposed. By dividing the integral interval symmetrically into two parts and applying the NJSC method symmetrically to the two weakly singular FIEs respectively, the mild singularities of the interval endpoints can be captured and the exponential convergence can be obtained. A detailed L^∞ convergence analysis of the numerical solution is derived. The NJSC method is then extended to two dimensional case and similar exponential convergence results are obtained for low regular solutions. Numerical examples are presented to demonstrate the efficiency of the proposed method.

AMS subject classifications: 65L70, 45B05

Key words: Nonpolynomial Jacobi spectral-collocation method, Fredholm integral equations, weakly singular, exponential convergence.

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1 Introduction

Fredholm integral equations of the second kind often arise in practical applications such as astrophysics, mathematical problems of radiative equilibrium, electrical engineering and radiative heat transfer problems [1,35–37]. In this paper, we consider weakly singular Fredholm integral equations (FIEs) of the second kind

$$u(t)=g(t)+\int_I k(t,s)u(s)ds, \quad t\in I=[0,1], \tag{1.1}$$

where the function $g(t)\in C(I)$, $u(t)$ is the solution to be determined, the kernel function $k(t,s)$ is weakly singular.

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The weakly singular FIE of the second kind usually has a solution with mild singularities at the boundary of solving domain. This integral can be discretized using generalized Gaussian quadratures [48] which take into account the nature of the integrand as well as the precise geometry of the interaction list. However, this approach is far from optimal in terms of the number of quadrature nodes needed for achieving a pre-selected desirable precision in higher dimensions [29]. We would like to remark here the recursively compressed inverse preconditioning (RCIP) method, as an efficient and accurate treatment at corner singularities based on the Nyström method [14], albeit with a slightly lower accuracy. In order to deal with the mild singularities, another frequently used technology is the graded mesh and there are considerable excellent relevant research works such as [13, 20, 36, 43]. However, the use of graded mesh has a practical limitation which serious round off errors occur when the initial stepsize becomes very small. To avoid this problem, many other methods have been proposed. For example, Tang [41] used spline collocation methods to solve Volterra integro-differential equations with weakly singular kernels by suitable graded mesh and obtained the optimal convergence. Cao and Xu [7] developed a singularity preserving Galerkin method in which some non-polynomial functions reflecting the singularity of the exact solution and quasi-uniform partitions are used to avoid round-off errors. Furthermore, Cao et al. [6] presented numerical solutions of weakly singular FIEs of the second kind by hybrid collocation method that preserves the singularity of the exact solution and at the same time provides the optimal order of convergence. Wang [44] further developed hybrid multistep collocation method for the weakly singular FIEs, which converges faster with lower degrees of freedom and more efficiently captures the weakly singular properties by nonpolynomial interpolation at the first subinterval. Smoothing transformation that transform the current weakly singular solution to a smoother one is also a popular strategy, see, for example [25, 30–32]. As an efficient method, Galerkin method has been applied to solve Volterra integral equations with weakly singular kernels numerically. Yi and Guo [49] presented an h-p version of the continuous Petrov-Galerkin (CPG) finite element method for linear Volterra integro-differential equations with smooth and nonsmooth kernels. Wang et al. [46] developed and analyzed an hp-version of the discontinuous Galerkin time-stepping method for linear Volterra integral equations with weakly singular kernels. Yi et al. [50] proposed a very simple but efficient postprocessing technique for improving the global accuracy of the discontinuous Galerkin (DG) time stepping method for solving nonlinear Volterra integro-differential equations by adding a higher order generalized Jacobi polynomial of degree $k+1$ with parameters $(-1,0)$ to the DG approximation of degree k. In addition, postprocessing method, including interpolation postprocessing and iteration postprocessing, is a common accelerated convergence technique. Huang and Zhang [19] discussed the superconvergence of the interpolated collocation solutions for Hammerstein equations. Then Huang and Wang [18] discussed the superconvergence of the interpolated postprocessing method for weakly singular Volterra integral equations (VIEs) of the second kind based on collocation method and hybrid collocation method. Graham [12] obtained the numerical solutions by the collocation and iterated collocation methods of the two-dimensional FIEs with weakly singular kernels.

Spectral method has been proven to be one of the most efficient approaches for solving integral equations and partial differential equations when the solution is sufficiently smooth. There are considerable research works with respect to the spectral method. For example, Chen et al. [8, 11] developed Jacobi-collocation spectral method for weakly singular Volterra integral equations with smooth solutions. Yang et al. [47] used this method to obtain the numerical solution of the weakly singular FIEs of the second kind with smooth solutions. Li and Tang [23, 24] presented Jacobi spectral collocation method for Abel-Volterra integral equations of the second kind and obtained the numerical solutions for weakly singular Volterra integral equations by Chebyshev and Legendre pseudo-spectral Galerkin methods. Sheng et al. [40] presented a multistep Legendre-Gauss spectral collocation method for nonlinear Volterra integral equations. Shen et al. [38] proposed generalized Jacobi spectral-Galerkin method for nonlinear weakly singular Volterra integral equations. Huang and Stynes [17] applied spectral collocation method to solve weakly singular Volterra integral equation of the second kind with a non-smooth solution and showed that the iterated spectral collocation solution is more accurate. Zhang et al. [27] investigated the hp spectral element approximation for optimal control problem governed by elliptic equation with an integral constraint for state. Kant et al. [21] proposed the Jacobi spectral Galerkin and Jacobi spectral multi Galerkin methods with their iterated versions for obtaining the superconvergence results of a general class of nonlinear Volterra integral equations with an Abel-type kernel and an endpoint singularity. Chen et al. [26] used the piecewise spectral collocation method to solve the second order Volterra integral differential equation with nonvanishing delay and the convergence of the spectral collocation method is proved by the Dirichlet formula. By the first-order optimality condition consisting of a Lagrange multiplier, Zhang et al. [28] designed a spectral Galerkin discrete scheme with weighted orthogonal Jacobi polynomials to approximate the integral state-constrained fractional optimal control problems and discussed a priori error estimates for state, adjoint state and control variables. Although it is of wide application, the traditional spectral methods which depend crucially on the regularity of the solution will lose exponential accuracy when facing problems with solutions of limited regularity. We take [47] for example, when the polynomial degree $N=6$, the L^∞ error of the numerical solution is 10^{-16} with smooth solution but only 10^{-1} with non-smooth solution. To overcome the exponential accuracy losing caused by the low regularity solution, many nonpolynomial spectral methods which may well catch the singularity of the solutions are proposed in recent years. Then the integral term in the resulting VIE can be approximated by Gauss quadrature formulas using the Chebyshev collocation points. Hou et al. [15, 16] proposed Müntz spectral method and applied this nonpolynomial Jacobi spectral-collocation method to the weakly singular Volterra integral equations of the second kind and integro-differential equations and fractional differential equations, and showed the exponential convergence rate can be achieved for solutions which are smooth after the variable change $t \to t^{\frac{1}{\lambda}}$ for a suitable real number λ. Chen and Shen [9, 10] introduced log orthogonal functions method and obtained the spectrally accurate approximation to subdiffusion equations.

However, the nonpolynomial Jacobi spectral-collocation (NJSC) method has not yet been applied to solve FIEs, probably because of the totally different non-local properties of the solutions compared with VIEs and the optimal choice of λ was not mentioned. In this paper, by equally dividing the integral interval into two parts for 1-dimensional problem, we implement the nonpolynomial Jacobi spectral-collocation method with symmetric nonpolynomial Jacobi functions on the two subintervals respectively and get the desired exponential convergence of the nonpolynomial Jacobi spectral-collocation solutions for weakly singular Fredholm integral equations of the second kind. We also give suggestions about the optimal choice of λ for this efficient method.

This paper is organized as follows. In Section 2, we present some preliminaries of nonpolynomial Jacobi functions and the approximation space. In Section 3, we describe the nonpolynomial Jacobi spectral-collocation method for weakly singular Fredholm integral equations of the second kind and analyze the L^∞ convergence error. In Section 4, we give suggestions about the selection of the optimal λ. Section 5 provides numerical examples to demonstrate the effectiveness of the nonpolynomial Jacobi spectral-collocation method.

2 Preliminaries

We first give the definition of nonpolynomial Jacobi function which is a class of fractional polynomial and their fundamental properties. These results play a key role in the convergence analysis of the proposed numerical method for (1.1).

For $I=[0,1]$, we introduce the nonpolynomial space

$$P_n^\lambda(I)=\text{span}\{1,t^\lambda,t^{2\lambda},\cdots,t^{n\lambda}\},$$

and the standard form of a nonpolynomial function $p_n^\lambda(t)\in P_n^\lambda(I)$ is as follows:

$$p_n^\lambda(t)=k_nt^{n\lambda}+k_{n-1}t^{(n-1)\lambda}+\cdots+k_1t^\lambda+k_0,\quad k_n\neq 0,\quad 0<\lambda\leq 1.$$

When $\lambda=1$, $P_n^1(I)$ denotes the normal n-degree polynomial space.

Definition 2.1. *Let $\omega(t)\in L^1(I)$ be a positive weight function. A sequence of λ-polynomials $\{p_n^\lambda\}_{n=0}^\infty$ with degree n is said to be orthogonal in $L^2_\omega(I)$ if*

$$(p_n^\lambda,p_m^\lambda)_\omega=\int_0^1 p_n^\lambda(t)p_m^\lambda(t)\omega(t)dt=\gamma_n\delta_{m,n},$$

where

$$\gamma_n=\|p_n^\lambda\|^2_{0,\omega}:=(p_n^\lambda,p_n^\lambda)_\omega,$$

and $\delta_{m,n}$ is the Kronecker delta. $L^2_\omega(I)$ is the L^2-weighted space.

It is well known that the classical Jacobi polynomial $J_n^{\alpha,\beta}(t)$, $t\in[-1,1]$ of degree n is a kind of orthogonal polynomial with weight function $\omega^{\alpha,\beta}(t)=(1-t)^{\alpha}(1+t)^{\beta}$ $(\alpha,\beta>-1)$ and the representation is as follows:

$$J_n^{\alpha,\beta}(t)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)}\Big(\frac{t-1}{2}\Big)^k.$$

We introduce the nonpolynomial Jacobi function of degree n on $I=[0,1]$,

$$J_n^{\alpha,\beta,\lambda}(t):=J_n^{\alpha,\beta}(2t^{\lambda}-1),\quad \forall t\in I, \tag{2.1}$$

with the weight function:

$$\omega^{\alpha,\beta,\lambda}(t)=\lambda(1-t^{\lambda})^{\alpha}t^{(\beta+1)\lambda-1}.$$

When $\lambda=1$, the functions $\{J_n^{\alpha,\beta,1}(t)\}_{n=0}^{\infty}$ are the classical Jacobi polynomials of I with the weight $\omega^{\alpha,\beta,1}(t)=(1-t)^{\alpha}t^{\beta}$.

For the positive weight function $\omega^{\alpha,\beta,\lambda}(t)\in L^1(I)$, $L^2_{\omega^{\alpha,\beta,\lambda}}(I)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta,\lambda}}(I)=\{v:\ v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta,\lambda}}<\infty\},$$

equipped with the norm

$$\|v\|_{\omega^{\alpha,\beta,\lambda}}=\Big(\int_I|v(t)|^2\omega^{\alpha,\beta,\lambda}(t)dt\Big)^{\frac{1}{2}}$$

and the inner product

$$(u,v)_{\omega^{\alpha,\beta,\lambda}}=\int_I u(t)v(t)\omega^{\alpha,\beta,\lambda}(t)dt,\quad \forall u,v\in L^2_{\omega^{\alpha,\beta,\lambda}}(I).$$

To measure the truncation error, we introduce the classical non-uniformly Jacobi-weighted Sobolev space $H^m_{\omega^{\alpha,\beta,1}}(I)$:

$$H^m_{\omega^{\alpha,\beta,1}}(I)=\{v:v^{(k)}\in L^2_{\omega^{\alpha+k,\beta+k,1}}(I),\ 0\le k\le m\},\quad m\in N,$$

equipped with the inner product, norm and semi norm as

$$(u,v)_{m,\omega^{\alpha,\beta,1}}=\sum_{k=0}^{m}(\partial_t^k u,\partial_t^k v)_{\omega^{\alpha+k,\beta+k,1}},\quad \|v\|_{m,\omega^{\alpha,\beta,1}}=(v,v)^{\frac{1}{2}}_{m,\omega^{\alpha,\beta,1}},$$
$$|v|_{m,\omega^{\alpha,\beta,1}}=\|\partial_t^m v\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

We denote the first order new derivative as:

$$D^1_{\lambda}v(t):=\frac{\partial_{\lambda}v(t)}{\partial t}=\frac{t^{1-\lambda}}{\lambda}\partial_t v(t).$$

When $\lambda=1$, $D_1^1 v(t)=\partial_t v(t)$, where $\partial_t v(t)$ is the first order classical derivative of $v(t)$,

$$D_\lambda^k v(t):=\underbrace{D_\lambda^1\cdots D_\lambda^1}_{k} v(t).$$

The non-uniformly nonpolynomial Jacobi-weighted Sobolev space is defined as

$$B_{\alpha,\beta}^{m,\lambda}(I)=\{v:\ D_\lambda^k v\in L^2_{\omega^{\alpha+k,\beta+k,\lambda}}(I),\ 0\le k\le m\}.$$

And the corresponding inner product, norm and semi-norm are as follows:

$$(u,v)_{B_{\alpha,\beta}^{m,\lambda}}=\sum_{k=0}^{m}\int_I D_\lambda^k u(t) D_\lambda^k v(t)\omega^{\alpha+k,\beta+k,\lambda}(t)dt,\quad \|v\|_{B_{\alpha,\beta}^{m,\lambda}}=(v,v)^{\frac{1}{2}}_{B_{\alpha,\beta}^{m,\lambda}},$$
$$|v|_{B_{\alpha,\beta}^{m,\lambda}}=\|D_\lambda^m v\|_{0,\omega^{\alpha+m,\beta+m,\lambda}}.$$

When $\lambda=1$, $B_{\alpha,\beta}^{m,\lambda}(I)=H^m_{\omega^{\alpha,\beta,1}}(I)=B_{\alpha,\beta}^{m,1}(I)$.

Remark 2.1. The definitions of $L^2_{\omega^{\alpha,\beta,\lambda}}(I)$, $H^m_{\omega^{\alpha,\beta,1}}(I)$ and $B_{\alpha,\beta}^{m,\lambda}(I)$ can be easily extended to the multi-dimensional spaces by the tensor product of 1-dimensional case.

Lemma 2.1 ([16]). *A function v belongs to $B_{\alpha,\beta}^{m,\lambda}(I)$ if and only if $v(t^{\frac{1}{\lambda}})$ belongs to $B_{\alpha,\beta}^{m,1}(I)$.*

Lemma 2.1 shows that a weakly singular function $v(t)$ becomes smoother under the new derivative. We take $\lambda=\frac{1}{2}$ and $v(t)=\sqrt{t}$ for example. In fact, $v(t)\in C(I)$, $v'(t)\notin C(I)$, but

$$D_\lambda^1 v(t)=\frac{t^{1-\frac{1}{2}}}{\frac{1}{2}}\cdot\frac{1}{2}t^{-\frac{1}{2}}=1,$$

which shows $v(t)\in B_{\alpha,\beta}^{\infty,\lambda}(I)$. On the other hand, we have

$$v(t^{\frac{1}{\lambda}})=t\in B_{\alpha,\beta}^{\infty,1}(I).$$

For $r\ge 0$ and $\kappa\in[0,1]$, let $C^{r,\kappa}(I)$ denotes the space of functions whose r-th derivatives are Hölder continuous with exponent κ, that is,

$$C^{r,\kappa}(I):=\left\{v\in C^r(I)\ \middle|\ d_\kappa\equiv\sup\frac{|v(t)-v(\tau)|}{|t-\tau|^\kappa}<\infty\right\}$$

endowed with the usual norm $\|\cdot\|_{r,\kappa}$

$$\|v\|_{r,\kappa}=\max_{0\le i\le r}\max_{t\in I}|\partial_t^i v(t)|+\max_{0\le i\le r}\sup_{t,\tau\in I,\ t\neq\tau}\frac{|\partial_t^i v(t)-\partial_t^i v(\tau)|}{|t-\tau|^\kappa}.$$

Then, there follows

Lemma 2.2 ([33, 34]). *Let r be a non-negative integer and $\kappa \in (0,1)$. Then for any $v \in C^{r,\kappa}(I)$, there exists a polynomial function $\mathcal{T}_N v$ from $C^{r,\kappa}(I)$ to $P_N^1(I)$ such that*

$$\|v - \mathcal{T}_N v\|_\infty \le C N^{-(r+\kappa)} \|v\|_{r,\kappa}.$$

Remark 2.2. Similarly, for two dimensional case, we also have the same result as Lemma 2.2.

3 Nonpolynomial Jacobi spectral-collocation method for the weakly singular FIE

In this section, we consider the one-dimensional weakly singular Fredholm integral equations of the second kind

$$u(t) = g(t) + \int_0^1 |s-t|^{-\gamma} k(t,s) u(s) ds, \quad t \in I = [0,1], \quad 0 < \gamma < 1, \tag{3.1}$$

where $g(t) \in C(I)$, $k(t,s) \in C(I \times I)$ with $K(t,t) \neq 0$ for $t \in I$, $u(t)$ is the solution to be determined.

The operator form of (3.1) is as follows,

$$u(t) = g(t) + (\mathcal{K}u)(t), \quad t \in I, \tag{3.2}$$

where $\mathcal{K}$ is the weakly singular integral operator defined by

$$(\mathcal{K}u)(t) = \int_0^1 |s-t|^{-\gamma} k(t,s) u(s) ds.$$

It is easy to verify that $\mathcal{K} : C(I) \to C(I)$ is compact. Assume that 1 is not an eigenvalue of $\mathcal{K}$, then (3.2) has unique solution in $C(I)$ (see [3]).

Unlike weakly singular VIEs, the weakly singular Fredholm integral equation has solutions with mild singularities at the two end-points 0 and 1, the nonpolynomial Jacobi spectral-collocation (NJSC) method therefore cannot directly be applied to this kind of integral equation as the application to weakly singular VIEs [16]. In this section, we divide the interval $[0,1]$ into two subintervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ and the relevant original FIE is rewritten into a system of two FIEs, which the weakly singularity only occurs at one of the end points. The NJSC method then can be extended to the FIE system after the variable changes. We then propose the numerical scheme of NJSC for FIEs of the second kind and the L^∞ convergence will be also analyzed.

3.1 Numerical scheme of Nonpolynomial Jacobi Spectral-Collocation method

To get the numerical scheme of (3.1), we divide $[0,1]$ into $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ such that the weakly singular points are divided into two subintervals which the weakly singularity only occurs at one of the end-points.

We introduce two sets of collocation points $\{t_n^1 := \frac{t_n}{2}\}_{n=1}^{N+1} \in [0,\frac{1}{2}]$ and $\{t_{N+2-n}^2 := 1 - \frac{t_n}{2}\}_{n=1}^{N+1} \in [\frac{1}{2},1]$, where $\{t_n\}_{n=1}^{N+1}$ are zeros in [0,1] of $J_{N+1}^{\alpha,\beta,\lambda}$. The relevant generalized basis functions of $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ with respect to the two sets of collocation points are as follows:

$$l_{j,\lambda}^{\alpha,\beta,1}(t) = \prod_{k=1,m\neq j}^{N+1} \frac{t^\lambda - (t_k^1)^\lambda}{(t_j^1)^\lambda - (t_k^1)^\lambda}, \qquad 0 \leq t \leq \frac{1}{2}, \qquad 1 \leq j \leq N+1, \tag{3.3a}$$

$$l_{j,\lambda}^{\alpha,\beta,2}(t) = \prod_{k=1,n\neq j}^{N+1} \frac{(1-t)^\lambda - (1-t_k^2)^\lambda}{(1-t_j^2)^\lambda - (1-t_k^2)^\lambda}, \qquad \frac{1}{2} \leq t \leq 1, \qquad 1 \leq j \leq N+1. \tag{3.3b}$$

Remark 3.1. $l_{j,\lambda}^{\alpha,\beta,2}(t)$ is obtained by the specular reflection transformation of $l_{j,\lambda}^{\alpha,\beta,1}(t)$ with respect to $t = \frac{1}{2}$. It can also be obtained by the nonpolynomial Lagrange interpolation formula based on the collocation points $\{t_{N+2-j}^2\}_{j=1}^{N+1}$, similar with $l_{j,\lambda}^{\alpha,\beta,1}(t)$.

For any $v(t) \in L^2_{\omega^{\alpha,\beta,\lambda}}(I)$, the generalized Jacobi-Gauss interpolation operator $I_{N,\lambda}^{\alpha,\beta}$ of degree N is defined as follows:

$$I_{N,\lambda}^{\alpha,\beta} v(t) = \begin{cases} I_{N,\lambda}^{\alpha,\beta,1} v(t) = \sum_{j=1}^{N+1} v(t_j^1) l_{j,\lambda}^{\alpha,\beta,1}(t), & t \in \left[0, \frac{1}{2}\right], \\ I_{N,\lambda}^{\alpha,\beta,2} v(t) = \sum_{j=1}^{N+1} v(t_j^2) l_{j,\lambda}^{\alpha,\beta,2}(t), & t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where t_j^k, $(k=1,2,\ j=1,2,\cdots,N+1)$ are the corresponding interpolate points in $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ respectively.

After dividing [0,1] into $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, we rewrite (3.1) into the following FIE system,

$$\begin{cases} u(t) - \left(\int_0^t + \int_t^{\frac{1}{2}} + \int_{\frac{1}{2}}^1\right) |t-s|^{-\gamma} k(t,s) u(s) ds = g(t), & t \in \left[0, \frac{1}{2}\right], \\ u(t) - \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^t + \int_t^1\right) |t-s|^{-\gamma} k(t,s) u(s) ds = g(t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{3.4}$$

The operator form of (3.4) is that

$$\begin{cases} u - (\mathcal{K}_1 u + \mathcal{K}_2 u + \mathcal{K}^1 u) = g, \\ u - (\mathcal{K}^2 u + \mathcal{K}_3 u + \mathcal{K}_4 u) = g. \end{cases} \tag{3.5}$$

where

$$(\mathcal{K}_1 u)(t) := \int_0^t |t-s|^{-\gamma} k(t,s) u(s) ds, \qquad (\mathcal{K}_2 u)(t) := \int_t^{\frac{1}{2}} |t-s|^{-\gamma} k(t,s) u(s) ds, \tag{3.6a}$$

$$(\mathcal{K}^1 u)(t) := \int_{\frac{1}{2}}^1 |t-s|^{-\gamma} k(t,s) u(s) ds, \qquad (\mathcal{K}^2 u)(t) := \int_0^{\frac{1}{2}} |t-s|^{-\gamma} k(t,s) u(s) ds, \tag{3.6b}$$

$$(\mathcal{K}_3 u)(t) := \int_{\frac{1}{2}}^t |t-s|^{-\gamma} k(t,s) u(s) ds, \qquad (\mathcal{K}_4 u)(t) := \int_t^1 |t-s|^{-\gamma} k(t,s) u(s) ds. \tag{3.6c}$$

The nonpolynomial Jacobi spectral-collocation method for (3.4) (or (3.5)) is to find nonpolynomial functions $u_N^\lambda(t) \in P_N^\lambda(I)$, such that

$$\begin{cases} u_N^\lambda(t_n^1) - \left(\int_0^{t_n^1} |t_n^1 - s|^{-\gamma} k(t_n^1, s) u_N^\lambda(s) ds + \int_{t_n^1}^{\frac{1}{2}} |t_n^1 - s|^{-\gamma} k(t_n^1, s) u_N^\lambda(s) ds \right. \\ \qquad \left. + \int_{\frac{1}{2}}^1 |t_n^1 - s|^{-\gamma} k(t_n^1, s) u_N^\lambda(s) ds \right) = g(t_n^1), \quad n = 1, \cdots, N+1, \\ u_N^\lambda(t_n^2) - \left(\int_0^{\frac{1}{2}} |t_n^2 - s|^{-\gamma} k(t_n^2, s) u_N^\lambda(s) ds + \int_{\frac{1}{2}}^{t_n^2} |t_n^2 - s|^{-\gamma} k(t_n^2, s) u_N^\lambda(s) ds \right. \\ \qquad \left. + \int_{t_n^2}^1 |t_n^2 - s|^{-\gamma} k(t_n^2, s) u_N^\lambda(s) ds \right) = g(t_n^2), \quad n = 1, \cdots, N+1. \end{cases} \tag{3.7}$$

The relevant operator form of (3.7) is as follows:

$$\begin{cases} u_N^\lambda - I_{N,\lambda}^{\alpha,\beta,1} (\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}^1) u_N^\lambda = I_{N,\lambda}^{\alpha,\beta,1} g, \\ u_N^\lambda - I_{N,\lambda}^{\alpha,\beta,2} (\mathcal{K}^2 + \mathcal{K}_3 + \mathcal{K}_4) u_N^\lambda = I_{N,\lambda}^{\alpha,\beta,2} g. \end{cases} \tag{3.8}$$

By using the variable changes

$$s = \begin{cases} t_n^1 \theta^{\frac{1}{\lambda}} := \tau_n^1(\theta), & s \in [0, t_n^1], \\ \dfrac{1}{2} - \left(\dfrac{1}{2} - t_n^1 \right) \theta^{\frac{1}{\lambda}} := \tau_n^2(\theta), & s \in \left[t_n^1, \dfrac{1}{2} \right], \\ \dfrac{1}{2} + \left(t_n^2 - \dfrac{1}{2} \right) \theta^{\frac{1}{\lambda}} := \tau_n^3(\theta), & s \in \left[\dfrac{1}{2}, t_n^2 \right], \\ 1 - (1 - t_n^2) \theta^{\frac{1}{\lambda}} := \tau_n^4(\theta), & s \in [t_n^2, 1], \end{cases}$$

to transform the integral intervals of the above six integrals into $(0,1)$, respectively. Then

(3.7) can be rewritten as follows:

$$\begin{cases} u_N^\lambda(t_n^1)-\bigg(\dfrac{(t_n^1)^{1-\gamma}}{\lambda}\displaystyle\int_0^1(1-\theta)^{-\gamma}\theta^{\frac{1}{\lambda}-1}\dfrac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^1,\tau_n^1(\theta))u_N^\lambda(\tau_n^1(\theta))d\theta \\ \quad+\dfrac{(\frac{1}{2}-t_n^1)^{1-\gamma}}{\lambda}\displaystyle\int_0^1(1-\theta)^{-\gamma}\theta^{\frac{1}{\lambda}-1}\dfrac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^1,\tau_n^2(\theta))u_N^\lambda(\tau_n^2(\theta))d\theta \\ \quad+\displaystyle\int_{\frac{1}{2}}^1|t_n^1-s|^{-\gamma}k(t_n^1,s)u_N^\lambda(s)ds\bigg)=g(t_n^1), \\ u_N^\lambda(t_n^2)-\bigg(\displaystyle\int_0^{\frac{1}{2}}|t_n^2-s|^{-\gamma}k(t_n^2,s)u_N^\lambda(s)ds \\ \quad+\dfrac{(t_n^2-\frac{1}{2})^{1-\gamma}}{\lambda}\displaystyle\int_0^1(1-\theta)^{-\gamma}\theta^{\frac{1}{\lambda}-1}\dfrac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^2,\tau_n^3(\theta))u_N^\lambda(\tau_n^3(\theta))d\theta \\ \quad+\dfrac{(1-t_n^2)^{1-\gamma}}{\lambda}\displaystyle\int_0^1(1-\theta)^{-\gamma}\theta^{\frac{1}{\lambda}-1}\dfrac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^2,\tau_n^4(\theta))u_N^\lambda(\tau_n^4(\theta))d\theta\bigg)=g(t_n^2), \\ \quad n=1,\cdots,N+1. \end{cases} \tag{3.9}$$

We approximate the above four weakly singular integrals whose corresponding integral operators $\{\mathcal{K}_i\}_{i=1}^4$ by Jacobi-Gauss numerical quadrature corresponding the discretized integral operators $\{\mathcal{K}_N^i\}_{i=1}^4$ with the weight function

$$w^{-\gamma,\frac{1}{\lambda}-1,1}(\theta)=(1-\theta)^{-\gamma}\theta^{\frac{1}{\lambda}-1}.$$

The rest two smooth integrals whose corresponding integral operators $\{\mathcal{K}^i\}_{i=1}^2$ are approximated by normal Legendre-Gauss quadrature $\{\mathcal{K}^{i,N}\}_{i=1}^2$ without affecting the whole accuracy of the numerical scheme. Then, the discretized nonpolynomial Jacobi spectral-collocation method is to find $\hat{u}_N^\lambda(t)\in P_N^\lambda(I)$:

$$\hat{u}_N^\lambda(t)=\begin{cases} \hat{u}_N^{\lambda,1}(t)=\displaystyle\sum_{i=1}^{N+1}\hat{U}_i^1 l_{i,\lambda}^{\alpha,\beta,1}(t),\ \ t\in\Big[0,\frac{1}{2}\Big], \\ \hat{u}_N^{\lambda,2}(t)=\displaystyle\sum_{i=1}^{N+1}\hat{U}_i^2 l_{i,\lambda}^{\alpha,\beta,2}(t),\ \ t\in\Big[\frac{1}{2},1\Big], \end{cases} \tag{3.10}$$

satisfying the following algebraic system is satisfied:

$$\begin{cases} \sum_{i=1}^{N+1}\hat{U}_i^1 l_i^1(t_n^1) - \Big(\sum_{i=1}^{N+1}\hat{U}_i^1\Big(\frac{(t_n^1)^{1-\gamma}}{\lambda}\sum_{p=1}^{N+1} w_p \frac{(1-\theta_p^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta_p)^{-\gamma}} k(t_n^1,\tau_n^1(\theta_p)) l_i^1(\tau_n^1(\theta_p)) \\ \quad + \frac{(\frac{1}{2}-t_n^1)^{1-\gamma}}{\lambda}\sum_{p=1}^{N+1} w_p \frac{(1-\theta_p^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta_p)^{-\gamma}} k(t_n^1,\tau_n^2(\theta_p)) l_i^1(\tau_n^2(\theta_p))\Big) \\ \quad + \sum_{i=1}^{N+1}\hat{U}_i^2 \sum_{p=1}^{N+1}\omega_p \Big|t_n^1 - \frac{\xi_p+3}{4}\Big|^{-\gamma} k\Big(t_n^1,\frac{\xi_p+3}{4}\Big) l_i^2\Big(\frac{\xi_p+3}{4}\Big)\Big) = g(t_n^1), \\ \sum_{i=1}^{N+1}\hat{U}_i^2 l_i^2(t_n^2) - \Big(\sum_{i=1}^{N+1}\hat{U}_i^1 \sum_{p=1}^{N+1}\omega_p \Big|t_n^2 - \frac{\xi_p+1}{4}\Big|^{-\gamma} k\Big(t_n^2,\frac{\xi_p+1}{4}\Big) l_i^1\Big(\frac{\xi_p+1}{4}\Big) \\ \quad + \sum_{i=1}^{N+1}\hat{U}_i^2\Big(\frac{(t_n^2-\frac{1}{2})^{1-\gamma}}{\lambda}\sum_{p=1}^{N+1} w_p \frac{(1-\theta_p^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta_p)^{-\gamma}} k(t_n^2,\tau_n^3(\theta_p)) l_i^2(\tau_n^3(\theta_p)) \\ \quad + \frac{(1-t_n^2)^{1-\gamma}}{\lambda}\sum_{p=1}^{N+1} w_p \frac{(1-\theta_p^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta_p)^{-\gamma}} k(t_n^2,\tau_n^4(\theta_p)) l_i^2(\tau_n^4(\theta_p))\Big)\Big) = g(t_n^2), \end{cases} \tag{3.11}$$

where $\{\theta_p\}_{p=1}^{N+1}$ are the zeros of $J_{N+1}^{-\gamma,1/(\lambda-1),1}$, $\{w_p\}_{p=1}^{N+1}$ are the corresponding weights. $\{\xi_p\}_{p=1}^{N+1}$ are the Jacobi-Gauss quadrature nodes in $[-1,1]$, $\{\omega_p\}_{p=1}^{N+1}$ are the corresponding weights. $\hat{U}_i^1$, $\hat{U}_i^2$, $i=1,\cdots,N+1$ are the unknowns to be determined.

The operator form of the discretized NJSC equations (3.11) in $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ is as follows

$$\begin{cases} \hat{u}_N^{\lambda,1} - I_{N,\lambda}^{\alpha,\beta,1}(\mathcal{K}_N^1+\mathcal{K}_N^2)\hat{u}_N^{\lambda,1} + I_{N,\lambda}^{\alpha,\beta,1}\mathcal{K}^{1,N}\hat{u}_N^{\lambda,2} = I_{N,\lambda}^{\alpha,\beta,1} g, \\ \hat{u}_N^{\lambda,2} - I_{N,\lambda}^{\alpha,\beta,2}\mathcal{K}^{2,N}\hat{u}_N^{\lambda,1} + I_{N,\lambda}^{\alpha,\beta,2}(\mathcal{K}_N^3+\mathcal{K}_N^4)\hat{u}_N^{\lambda,2} = I_{N,\lambda}^{\alpha,\beta,2} g. \end{cases} \tag{3.12}$$

Remark 3.2. The solving idea of the above linear weakly singular FIE by the nonpolynomial Jacobi spectral-collocation method can be easily extended to the nonlinear weakly singular FIE case. For the nonlinear equation, we finally obtain a nonlinear system and Newton's iteration method is usually used to get the unknown coefficients. However, at each iteration, the integrals will be evaluated numerically and the computation complexity will increase. In the future work, we will construct a new nonpolynomial Jacobi spectral-collocation method to solve this problem to reduce the computation complexity, in which those integrals can be evaluated once just in the linear case.

3.2 L^∞ convergence analysis

In this subsection, some useful lemmas are given and the L^∞ convergence of the discretized nonpolynomial Jacobi spectral collocation solution is analyzed. First, we present the error estimation for the nonpolynomial Jacobi-Gauss interpolation operator based on the roots of $J^{\alpha,\beta,\lambda}_{N+1}(t)$ in the following lemma.

Lemma 3.1 ([16]). *If* $-1<\alpha,\beta\le -\frac{1}{2}$, *for* $\forall v(t^{\frac{1}{\lambda}})\in B^{m,1}_{\alpha,\beta}(I)$, *we have*

$$\|v-I^{\alpha,\beta,k}_{N,\lambda}v\|_\infty \le cN^{\frac{1}{2}-m}\|\partial^m_t v(t^{\frac{1}{\lambda}})\|_{0,\omega^{\alpha+m,\beta+m,1}}, \quad m\ge 1, \quad k=1,2.$$

The following lemma gives the error estimation of the numerical quadrature for the smooth integral.

Lemma 3.2 ([11]). *If* $v\in H^m_{\omega^{\alpha,\beta,1}}(I)$ *and* $\phi\in P^1_N(I)$, *then for the Jacobi Gauss and Jacobi Gauss-Radau integration we have*

$$|(v,\phi)_{\omega^{\alpha,\beta,1}}-(v,\phi)_N|\le CN^{-m}\|D^m_1 v\|_{0,\omega^{\alpha+m,\beta+m,1}}\|\phi\|_{0,\omega^{\alpha,\beta,1}},$$

where

$$(v,\phi)_N=\sum_{j=1}^{N+1} v(t_j)\phi(t_j)\omega_j,$$

$\{t_j\}_{j=1}^{N+1}$ *are Jacobi Gauss points or Jacobi Gauss-Radau points and* $\{\omega_j\}_{j=1}^{N+1}$ *are the corresponding weights.*

The following result presents the error estimation of numerical quadrature for the weakly singular integral.

Lemma 3.3 ([16]). *For all* $v\in B^{m,1}_{\alpha,\beta}(I)$, $m\ge 1$ *and* $\phi\in P^1_N(I)$, *we have*

$$|(v,\phi)_{\omega^{\alpha,\beta,1}}-(v,\phi)_{N,\omega^{\alpha,\beta,1}}|\le CN^{-m}\|D^m_1 v\|_{0,\omega^{\alpha+m,\beta+m,1}}\|\phi\|_{0,\omega^{\alpha,\beta,1}},$$

where

$$(v,\phi)_{N,\omega^{\alpha,\beta,1}}=\sum_{j=1}^{N+1} v(\theta_j)\phi(\theta_j)\omega^{\alpha,\beta,1}_j,$$

$\{\theta_j\}_{j=1}^{N+1}$ *are the zeros of Jacobi polynomial* $J^{\alpha,\beta,1}_{N+1}$, *and* $\{\omega^{\alpha,\beta,1}_j\}_{j=1}^{N+1}$ *are the corresponding weights.*

The above Lemma 3.3 shows the case $N+1\ge m$ of Lemma 2.9 in [16]. See [4, 39] for further details.

Lemma 3.4 ([16]). *For any function* $v(t)\in C(I)$, $k\in C(I\times I)$ *and* $0<\kappa<1-\gamma$, *the integral operator* $\mathcal{K}_1:\ C(I)\to C^{0,\kappa}[0,1]$ *and* $|t-s|^{-\gamma}k(\cdot,s)\in C^{0,\kappa}[0,1]$ *with* $0<\kappa<1-\gamma$, *there exists a positive constant c such that*

$$\frac{|(\mathcal{K}_1 v)(t^{\frac{1}{\lambda}})-(\mathcal{K}_1 v)(s^{\frac{1}{\lambda}})|}{|t-s|^\kappa}\le c\max_{t\in I}|v(t)|, \quad \forall t,s\in I, \quad t\ne s.$$

Thus,

$$\|(\mathcal{K}_1 v)(t^{\frac{1}{\lambda}})\|_{0,\kappa}\le c\|v\|_\infty.$$

From Lemma 3.4, it is easy to obtain similar results for $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$:

$$\|(\mathcal{K}_i v)(t^{\frac{1}{\lambda}})\|_{0,\kappa} \le c\|v\|_\infty, \quad i=2,3,4.$$

Lemma 3.5 ([16]). *Let $\{l_{j,\lambda}^{\alpha,\beta,k}(t)\}_{j=0}^N$ $(k=1,2)$ be generalized Lagrange interpolation basis functions associated with the Gauss points of the nonpolynomial Jacobi function $J_{N+1}^{\alpha,\beta,\lambda}(t)$. Then,*

$$\|I_{N,\lambda}^{\alpha,\beta,k}\|_\infty := \max_{t\in I}\sum_{j=0}^N |l_{j,\lambda}^{\alpha,\beta,k}(t)| = \begin{cases} \mathcal{O}(\log N), & -1<\alpha,\beta\le -\dfrac{1}{2}, \\ \mathcal{O}(N^{\gamma+\frac{1}{2}}), \quad \gamma=\max(\alpha,\beta), & \textit{otherwise}. \end{cases}$$

Based on the above lemmas, we give the error estimate of the proposed NJSC method.

Theorem 3.1. *Let $u(t)$ be the exact solution to Eq.* (3.1) *and $\hat{u}_N^\lambda(t)$ be the solution of the discretized nonpolynomial Jacobi spectral-collocation equation* (3.11)*. Assume $0<\gamma<1$, $-1<\alpha,\beta\le -\frac{1}{2}$, $k(t,s)\in C^m(I,I)$ and $u(t^{\frac{1}{\lambda}})\in B_{\alpha,\beta}^{m,1}(I)$, $m\ge 1$. Then we have*

$$\|u-\hat{u}_N^\lambda\|_\infty \le cN^{\frac{1}{2}-m}\left(\|D_1^m u(t^{\frac{1}{\lambda}})\|_{0,\omega^{\alpha+m,\beta+m,1}} + N^{-\frac{1}{2}}\log N k^*\|u\|_\infty\right), \tag{3.13}$$

where k^* is a constant independent of N defined in the following proof.

Proof. Let

$$e(t) = \begin{cases} u(t)-\hat{u}_N^{\lambda,1}(t) = e_1(t), & t\in\left[0,\dfrac{1}{2}\right], \\ u(t)-\hat{u}_N^{\lambda,2}(t) = e_2(t), & t\in\left(\dfrac{1}{2},1\right], \end{cases}$$

be the error function and noting that $I_{\lambda,N}^{\alpha,\beta,i}\mathcal{K}u(t_n^i) = \mathcal{K}u(t_n^i)$, $i=1,2$. It is easy to see that $u(t)$ also satisfies (3.8). Substituted $u_N^\lambda(t)$ by $u(t)$ in (3.8) and subtracting it from (3.12), we can get for $n=1,\cdots,N+1$,

$$\begin{cases} e_1(t_n^1) = (\mathcal{K}_1+\mathcal{K}_2)u(t_n^1) - \left(\mathcal{K}_N^1+\mathcal{K}_N^2\right)u_N^\lambda(t_n^1) + \mathcal{K}^1 u(t_n^1) - \mathcal{K}^{1,N}u_N^\lambda(t_n^1), & \text{(3.14a)} \\ e_2(t_n^2) = \mathcal{K}^2 u(t_n^2) - \mathcal{K}^{2,N}\hat{u}_N^\lambda(t_n^2) + (\mathcal{K}_3+\mathcal{K}_4)u(t_n^2) - \left(\mathcal{K}_N^3+\mathcal{K}_N^4\right)u_N^\lambda(t_n^2). & \text{(3.14b)} \end{cases}$$

Multiplying both sides of (3.14a) by $l_{n,\lambda}^{\alpha,\beta,1}(t)$ and summing up the resulting equation from $n=1$ to $N+1$ gives

$$\begin{aligned} I_{\lambda,N}^{\alpha,\beta,1}u(t) - \hat{u}_N^{\lambda,1}(t) =& I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}_1+\mathcal{K}_2)u(t) - \sum_{n=1}^{N+1}\left[\left(\mathcal{K}_N^1+\mathcal{K}_N^2\right)\hat{u}_N^{\lambda,1}(t_n^1)\right]l_{n,\lambda}^{\alpha,\beta,1}(t) \\ &+ I_{\lambda,N}^{\alpha,\beta,1}\mathcal{K}^1 u(t) - \sum_{n=1}^{N+1}\mathcal{K}^{1,N}\hat{u}_N^{\lambda,2}(t_n^1)l_{n,\lambda}^{\alpha,\beta,1}(t) \\ =& I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}_1+\mathcal{K}_2)e_1(t) + I_{\lambda,N}^{\alpha,\beta,1}((\mathcal{K}_1+\mathcal{K}_2)-(\mathcal{K}_N^1+\mathcal{K}_N^2))\hat{u}_N^{\lambda,1}(t) \\ &+ I_{\lambda,N}^{\alpha,\beta,1}\mathcal{K}^1 e_2(t) I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}^1-\mathcal{K}^{1,N})\hat{u}_N^{\lambda,2}(t). \end{aligned} \tag{3.15}$$

By (3.15), we have

$$\begin{aligned}
e_1(t)=&u(t)-I_{\lambda,N}^{\alpha,\beta,1}u(t)+I_{\lambda,N}^{\alpha,\beta,1}u(t)-\hat{u}_N^{\lambda,1}(t)\\
=&u(t)-I_{\lambda,N}^{\alpha,\beta,1}u(t)+I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}_1+\mathcal{K}_2)e_1(t)+I_{\lambda,N}^{\alpha,\beta,1}((\mathcal{K}_1+\mathcal{K}_2)-(\mathcal{K}_N^1+\mathcal{K}_N^2))\hat{u}_N^{\lambda,1}(t)\\
&+I_{\lambda,N}^{\alpha,\beta,1}\mathcal{K}^1e_2(t)+I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}^1-\mathcal{K}^{1,N})\hat{u}_N^{\lambda,2}(t)\\
=&\underbrace{u(t)-I_{\lambda,N}^{\alpha,\beta,1}u(t)}_{I_1^1}+\underbrace{I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}_1+\mathcal{K}_2)e_1(t)-(\mathcal{K}_1+\mathcal{K}_2)e_1(t)}_{I_2^1}+(\mathcal{K}_1+\mathcal{K}_2)e_1(t)\\
&+\underbrace{I_{\lambda,N}^{\alpha,\beta,1}((\mathcal{K}_1+\mathcal{K}_2)-(\mathcal{K}_N^1+\mathcal{K}_N^2))\hat{u}_N^{\lambda,1}(t)}_{I_3^1}+\underbrace{I_{\lambda,N}^{\alpha,\beta,1}\mathcal{K}^1e_2(t)-\mathcal{K}^1e_2(t)}_{I_4^1}+\mathcal{K}^1e_2(t)\\
&+\underbrace{I_{\lambda,N}^{\alpha,\beta,1}(\mathcal{K}^1-\mathcal{K}^{1,N})\hat{u}_N^{\lambda,2}(t)}_{I_5^1}\\
=&(\mathcal{K}_1+\mathcal{K}_2)e_1(t)+\mathcal{K}^1e_2(t)+I_1^1+I_2^1+I_3^1+I_4^1+I_5^1,
\end{aligned}$$

with corresponding respectively the interpolation error (I_1^1, I_2^1, I_4^1), and the numerical quadrature error (I_3^1, I_5^1).

Similarly, multiplying both sides of (3.14b) by $l_{n,\lambda}^{\alpha,\beta,2}(t)$ and summing up the resulting equation from $n=1$ to $N+1$ gives,

$$e_2(t)=u(t)-\hat{u}_N^{\lambda,2}(t)=((\mathcal{K}_3+\mathcal{K}_4)e_2)(t)+\mathcal{K}^2e_1(t)+I_1^2+I_2^2+I_3^2+I_4^2+I_5^2,$$

where

$$\begin{aligned}
&I_1^2=u(t)-I_{\lambda,N}^{\alpha,\beta,2}u(t),\quad I_2^2=I_{\lambda,N}^{\alpha,\beta,2}((\mathcal{K}_3+\mathcal{K}_4)e_2)(t)-((\mathcal{K}_3+\mathcal{K}_4)e_2)(t),\\
&I_3^2=I_{\lambda,N}^{\alpha,\beta,2}((\mathcal{K}_3+\mathcal{K}_4)-(\mathcal{K}_N^3-\mathcal{K}_N^4))\hat{u}_N^{\lambda,2}(t),\quad I_4^2=I_{\lambda,N}^{\alpha,\beta,2}\mathcal{K}^2e_1(t)-\mathcal{K}^2e_1(t),\\
&I_5^2=I_{\lambda,N}^{\alpha,\beta,2}(\mathcal{K}^2-\mathcal{K}^{2,N})\hat{u}_N^{\lambda,1}(t).
\end{aligned}$$

Therefore, we have

$$\begin{pmatrix}e_1(t)\\e_2(t)\end{pmatrix}=\begin{pmatrix}\mathcal{K}_1+\mathcal{K}_2, & \mathcal{K}^1\\ \mathcal{K}^2, & \mathcal{K}_3+\mathcal{K}_4\end{pmatrix}\begin{pmatrix}e_1(t)\\e_2(t)\end{pmatrix}+\begin{pmatrix}I_1^1+I_2^1+I_3^1+I_4^1+I_5^1\\I_1^2+I_2^2+I_3^2+I_4^2+I_5^2\end{pmatrix}\tag{3.16}$$

and

$$\left(I-\begin{pmatrix}\mathcal{K}_1+\mathcal{K}_2, & \mathcal{K}^1\\ \mathcal{K}^2, & \mathcal{K}_3+\mathcal{K}_4\end{pmatrix}\right)\begin{pmatrix}e_1(t)\\e_2(t)\end{pmatrix}=\begin{pmatrix}I_1^1+I_2^1+I_3^1+I_4^1+I_5^1\\I_1^2+I_2^2+I_3^2+I_4^2+I_5^2\end{pmatrix}.\tag{3.17}$$

Since $\{\mathcal{K}_i\}_{i=1}^4$ and $\{\mathcal{K}^i\}_{i=1}^2$ are compact operators and 1 is not an eigenvalue of $\begin{pmatrix}\mathcal{K}_1+\mathcal{K}_2, & \mathcal{K}^1\\ \mathcal{K}^2, & \mathcal{K}_3+\mathcal{K}_4\end{pmatrix}$, then $\left(I-\begin{pmatrix}\mathcal{K}_1+\mathcal{K}_2, & \mathcal{K}^1\\ \mathcal{K}^2, & \mathcal{K}_3+\mathcal{K}_4\end{pmatrix}\right)$ is invertible. By Fredholm Alternative theorem, we then obtain

$$\|e\|_\infty=\max\{\|e_1\|_\infty,\|e_2\|_\infty\}\le c\max\left\{\begin{matrix}\|I_1^1\|_\infty+\|I_2^1\|_\infty+\|I_3^1\|_\infty+\|I_4^1\|_\infty+\|I_5^1\|_\infty,\\ \|I_1^2\|_\infty+\|I_2^2\|_\infty+\|I_3^2\|_\infty+\|I_4^2\|_\infty+\|I_5^2\|_\infty\end{matrix}\right\}.\tag{3.18}$$

For I_1^1, it follows from Lemma 3.1 that,

$$\|I_1^1\|_\infty \le cN^{\frac{1}{2}-m}\|D_1^m u(t^{\frac{1}{\lambda}})\|_{0,\omega^{\alpha+m,\beta+m,1}}. \tag{3.19}$$

For I_2^1, let $t=z^{\frac{1}{\lambda}}$ and follows from Lemmas 2.2, 3.4 and 3.5, we obtain,

$$\begin{aligned}\|I_2^1\|_\infty =&\max_{t\in I}\Big|I_{\lambda,N}^{\alpha,\beta,1}\Big((\mathcal{K}_1+\mathcal{K}_2)e_1\Big)(t)-\Big((\mathcal{K}_1+\mathcal{K}_2)e_1\Big)(t)\Big|\\ =&\max_{t=z^{\frac{1}{\lambda}}\in I}\big|(I_{1,N}^{\alpha,\beta,1}-I)(\mathcal{K}_1e_1)(z^{\frac{1}{\lambda}})+(I_{1,N}^{\alpha,\beta,1}-I)(\mathcal{K}_2e_1)(z^{\frac{1}{\lambda}})\big|\\ =&\big\|(I_{1,N}^{\alpha,\beta,1}-I)[(\mathcal{K}_1e_1)(z^{\frac{1}{\lambda}})-\mathcal{T}_N(\mathcal{K}_1e_1)(z^{\frac{1}{\lambda}})]\\ &\quad+(I_{1,N}^{\alpha,\beta,1}-I)[(\mathcal{K}_2e_1)(z^{\frac{1}{\lambda}})-\mathcal{T}_N(\mathcal{K}_2e_1)(z^{\frac{1}{\lambda}})]\big\|_\infty\\ \le&\Big(\|I_{1,N}^{\alpha,\beta,1}\|_\infty+1\Big)\Big(\big\|(\mathcal{K}_1e_1)(z^{\frac{1}{\lambda}})-\mathcal{T}_N(\mathcal{K}_1e_1)(z^{\frac{1}{\lambda}})\big\|_\infty\\ &\quad+\big\|(\mathcal{K}_2e_1)(z^{\frac{1}{\lambda}})-\mathcal{T}_N(\mathcal{K}_2e_1)(z^{\frac{1}{\lambda}})\big\|_\infty\Big)\\ \le& cN^{-\kappa}\log N\|e_1\|_\infty,\quad 0<\kappa<1-\gamma. \end{aligned}\tag{3.20}$$

For I_3^1, we can obtain by Lemma 3.3 together with Lemma 3.5,

$$\begin{aligned}\|I_3^1\|_\infty =&\max_{t\in[0,1]}\Big|I_{\lambda,N}^{\alpha,\beta,1}\Big((\mathcal{K}_1+\mathcal{K}_2)-(\mathcal{K}_N^1+\mathcal{K}_N^2)\Big)\hat{u}_N^{\lambda,1}(t)\Big|\\ \le& c\|I_{\lambda,N}^{\alpha,\beta,1}\|_\infty\max_{t\in[0,1]}\Big|\Big((\mathcal{K}_1+\mathcal{K}_2)-(\mathcal{K}_N^1+\mathcal{K}_N^2)\Big)\hat{u}_N^{\lambda,1}(t)\Big|\\ \le& cN^{-m}\log N\max_{1\le n\le N+1}\|D_1^m\bar{k}(t_n^1,\theta)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}}\max_{1\le n\le N+1}\|\hat{u}_N^{\lambda,1}(\tau_n^1(\theta))\|_{0,\omega^{-\gamma,\frac{1}{\lambda}-1,1}}\\ \le& cN^{-m}\log N\max_{1\le n\le N+1}\|D_1^m\bar{k}(t_n^1,\theta)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}}\|\hat{u}_N^{\lambda,1}\|_\infty\\ \le& cN^{-m}\log N\max_{1\le n\le N+1}\|D_1^m\bar{k}(t_n^1,\theta)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}}(\|e_1\|_\infty+\|u\|_\infty), \end{aligned}\tag{3.21}$$

where

$$\bar{k}(t_n^1,\theta)=\frac{t_n^{1-\gamma}}{\lambda}\frac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^1,\tau_n^1(\theta))+\frac{(\frac{1}{2}-t_n^1)^{1-\gamma}}{\lambda}\frac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^1,\tau_n^2(\theta)).$$

For I_4^1, similarly with I_2^1, we can obtain:

$$\begin{aligned}\|I_4^1\|_\infty =&\max_{t\in I}\big|I_{\lambda,N}^{\alpha,\beta,1}\mathcal{K}^1e_2(t)-\mathcal{K}^1e_2(t)\big|\\ \le& cN^{-\kappa}\log N\|e_2\|_\infty,\quad 0<\kappa<1-\gamma. \end{aligned}\tag{3.22}$$

Similarly with $\|I_3^1\|_\infty$, we can obtain the estimate of $\|I_5^1\|_\infty$ by the Gauss quadrature error Lemma 3.2 together with Lemma 3.5

$$\|I_5^1\|_\infty\le cN^{-m}\log N\max_{1\le n\le N+1}\|D_1^m\hat{k}(t_n^1,\xi)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}}(\|e_2\|_\infty+\|u\|_\infty), \tag{3.23}$$

where

$$\hat{k}(t_n^1,\xi)=\Big|t_n^1-\frac{\xi+3}{4}\Big|^{-\gamma}k\Big(t_n^1,\frac{\xi+3}{4}\Big).$$

Similarly, we can get the estimation of

$$\|I_1^2\|_\infty+\|I_2^2\|_\infty+\|I_3^2\|_\infty+\|I_4^2\|_\infty+\|I_5^2\|_\infty.$$

Then we have:

$$\|e\|_\infty\le c\max_{t\in I}\left\{\begin{matrix} N^{\frac{1}{2}-m}\|D_1^m v(t^{\frac{1}{\lambda}})\|_{0,\omega^{\alpha+m,\beta+m,1}}+2N^{-\kappa}\log N\|e\|_\infty \\ +2N^{-m}\log N k_1^*(\|e\|_\infty+\|u\|_\infty), \\ N^{\frac{1}{2}-m}\|D_1^m v(t^{\frac{1}{\lambda}})\|_{0,\omega^{\alpha+m,\beta+m,1}}+2N^{-\kappa}\log N\|e\|_\infty \\ +2N^{-m}\log N k_2^*(\|e\|_\infty+\|u\|_\infty), \end{matrix}\right\}, \tag{3.24}$$

where

$$k_1^*=\max\Big\{\max_{1\le n\le N+1}\big\{\|D_1^m\hat{k}(t_n^1,\xi)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}},\|D_1^m\bar{k}(t_n^1,\theta)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}}\big\}\Big\}, \tag{3.25a}$$

$$k_2^*=\max\Big\{\max_{1\le n\le N+1}\big\{\|D_1^m\tilde{k}(t_n^2,\theta)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}},\|D_1^m\check{k}(t_n^2,\xi)\|_{0,\omega^{m-\gamma,m+\frac{1}{\lambda}-1,1}}\big\}\Big\}, \tag{3.25b}$$

$$\tilde{k}(t_n^2,\theta)=\frac{(t_n^2-\frac{1}{2})^{1-\gamma}}{\lambda}\frac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^2,\tau_n^3(\theta))+\frac{(1-t_n^2)^{1-\gamma}}{\lambda}\frac{(1-\theta^{\frac{1}{\lambda}})^{-\gamma}}{(1-\theta)^{-\gamma}}k(t_n^2,\tau_n^4(\theta)), \tag{3.25c}$$

$$\check{k}(t_n^2,\xi)=\Big|t_n^2-\frac{\xi_q+1}{4}\Big|^{-\gamma}k\Big(t_n^2,\frac{\xi+1}{4}\Big). \tag{3.25d}$$

Merging the estimation of (3.24), we complete the proof. □

Remark 3.3. The result of Theorem 3.1 can be straightforwardly extended to the case of the nonlinear weakly singular FIE under some suitable assumptions on the nonlinear term.

Remark 3.4. The nonpolynomial Jacobi spectral collocation method can be similarly extended to the two-dimensional weakly singular FIE of the second kind:

$$u(x,y)=g(x,y)+\int_0^1\int_0^1|x-t|^{-\gamma_1}|y-s|^{-\gamma_2}k(x,y,s,t)u(s,t)dtds, \tag{3.26}$$

where $(x,y)\in I\times I=\Omega=[0,1]\times[0,1]$, $0<\gamma_1,\gamma_2<1$, $g(x,y)\in C(\Omega)$, $k(s,t,x,y)\in C(\Omega\times\Omega)$, and $u(x,y)$ is the solution to be determined.

Specifically, we divide Ω into four subdomians and (3.26) can be rewrited into four equations. Then variable changes and numerical quadratures are applied to obtain the numerical schemes. The integral is divided into smooth integrals and unilaterally weakly singular integrals which are similar with weakly singular Volterra type integral. The smooth and weakly singular integrals are approximated by normal Legendre-Gauss and Jocabi-Gauss numerical quadratures with the weight function

$$w^{-\gamma,\frac{1}{\lambda}-1,1}(\theta)=(1-\theta)^{-\gamma}\theta^{\frac{1}{\lambda}-1}.$$

The L^∞ spectral error estimates can be obtained with similar process. We omit here and leave the proof to the reader.

4 The selection of λ

In this section, we give the suggestion about the optimal choice of λ for this nonpolynomial Jacobi spectral-collocation method. Since Fredholm integral equation can be splitted to Volterra integral equations with two integral terms. We take the following weakly singular Volterra integral equation of the second kind

$$u(t)-\int_0^t (t-s)^{-\gamma}k(s,t)u(s)ds=g(t), \quad t\in I=(0,1), \tag{4.1}$$

to show the optimal choice of λ. We know the following result.

Lemma 4.1 ([5]). *If $g(t)\in C^m(I)$ and $k\in C^m(\bar{I}\times\bar{I})$ with $k(s,s)\neq 0$, then the solution $u(t)$ of* (4.1) *can be expressed as*

$$u(t)=\sum_{(j,k)\in G}\gamma_{j,k}t^{j+k(1-\gamma)}+u_r(t), \tag{4.2}$$

where $G=\{(j,k): j,k$ are non-negative integers s.t. $j+k(1-\gamma)<m\}$, $\gamma_{j,k}$ are constant and $u_r(\cdot)\in C^m(\bar{I})$.

Theorem 4.1. *Let $u(t)$ be the exact solution to the Volterra integral equation* (4.1) *and $u_N^\lambda(t)$ is the solution of the discretized nonpolynomial Jacobi spectral-collocation equation. Assume $0<\gamma<1$, $-1<\alpha,\beta\leq -\frac{1}{2}$, which are parameters of nonpolynomial Jacobi function, $g(t)\in C^m(I)$, $k(s,t)\in C^m(\bar{I}\times\bar{I})$ with $m\geq 1$. Let $\lambda=(r_{jk})$, $(j,k)\in G$, where r_{jk} is defined as following* (4.3) *and $(\cdot)$ denotes the greatest common divisor, then the error $e=|u_N^\lambda(t)-u(t)|$ has the fastest rate of decay.*

Proof. As shown in (4.2), let

$$\hat{u}(t)=\sum_{(j,k)\in G}\gamma_{j,k}t^{j+k(1-\gamma)}.$$

Let

$$j+k(1-\gamma)=q_{jk}+r_{jk}, \tag{4.3}$$

where q_{jk} is an integer and $0<r_{jk}<1$, then

$$t^{j+k(1-\gamma)}=t^{q_{jk}}\cdot t^{r_{jk}}.$$

Let

$$\hat{u}_{jk}(t)=t^{r_{jk}} \quad \text{and} \quad \hat{u}(t)=\sum_{(j,k)\in G}\gamma_{j,k}t^{q_{jk}}\hat{u}_{jk}(t).$$

We take the derivative of $\hat{u}_{jk}(t)$

$$D^1_\lambda(\hat{u}_{jk}(t))=\frac{t^{1-\lambda}}{\lambda}r_{jk}t^{r_{kj}-1}=\frac{r_{jk}}{\lambda}t^{r_{jk}-\lambda}.$$

If $\lambda=r_{jk}$ or $r_{jk}-\lambda$ is an integer, $\hat{u}_{jk}(t)$ is sufficiently smooth under the new derivative. In view of $0<\lambda,r_{jk}<1$, we can conclude that $\lambda=r_{jk}$. Let $\hat{\lambda}=(r_{jk})$, $(j,k)\in G$, where $(a_1,a_2,\cdots,a_m)$ denotes the greatest common divisor of $a_1,a_2,\cdots,a_m$, then $\lambda=\hat{\lambda}$ is the optimal choice. □

Remark 4.1. Theorem 4.1 gives the suggestion of the optimal choice for λ. For example, if $\gamma=\frac{5}{6}$, then $1-\gamma=\frac{1}{6}$. According to the result above, r_{jk} should be $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$. $\frac{1}{6}$ is the greatest common divisor of them, therefore we take $\lambda=\frac{1}{6}$. If $\gamma=\frac{1}{\sqrt{2}}$ and $m=2$, r_{jk} should be $1-\frac{1}{\sqrt{2}}$, $2-\frac{2}{\sqrt{2}}$, $3-\frac{3}{\sqrt{2}}$. It is easy to see that $1-\frac{1}{\sqrt{2}}$ is "the greatest common divisor" of them, therefore we take $\lambda=1-\frac{1}{\sqrt{2}}$. The optimal λ can be equally applied to weakly singular Fredholm integral equation of the second kind in this paper.

5 Numerical results

In this section, numerical experiments are carried out to validate the convergence results proved in Section 3.

Example 5.1. Consider the following weakly singular FIE of the second kind

$$x(t)-\int_0^1|t-s|^{-\frac{1}{2}}x(s)ds=f(t),$$

where $f(t)$ is chosen such that the exact solution $x(t)=1+t^{\frac{1}{2}}+(1-t)^{\frac{1}{2}}$.

We compare the proposed NJSC method with the traditional spectral collocation method. In Fig. 1 the errors are given for different values of N. It can be seen that the L^∞ errors of the NJSC solution decay exponentially, but the errors of traditional spectral collocation solution decay algebraically.

Example 5.2. Consider the following linear weakly singular FIE of the second kind

$$5x(t)-\int_0^1|s-t|^{-\gamma}x(s)ds=f(t), \quad t\in(0,1).$$

We use

$$x^{(1)}(t)=t^{-\frac{2}{3}}\text{sint}\Big(\gamma=\frac{2}{3}\Big), \quad x^{(2)}(t)=(1-t)^{\frac{1}{2}}\Big(\gamma=\frac{1}{2}\Big), \quad x^{(3)}(t)=t^{\frac{2}{3}}+(1-t)^{\frac{2}{3}}\Big(\gamma=\frac{2}{3}\Big)$$

as the exact solutions with the right-hand sides $f(t)$ defined accordingly.

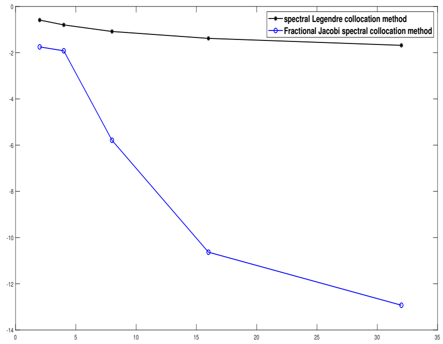

Figure 1: L^∞-norm errors of nonpolynomial Jacobi spectral collocation solution vs spectral Legendre collocation method

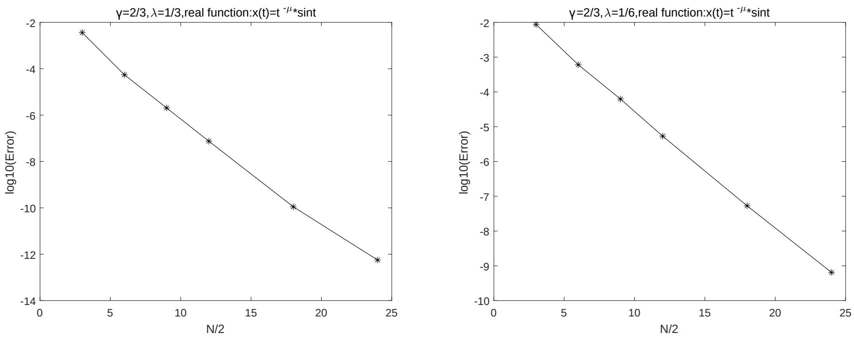

Figure 2: L^∞-norm errors for nonpolynomial Jacobi spectral collocation solutions for λ-polynomial degree N (DOF$=2N+2$) with exact solution $x^{(1)}(t)$.

The weakly singularity occurs at the left point 0 for $x^{(1)}(t)$, at the right point 1 for $x^{(2)}(t)$, and at the left and right points 0, 1 for $x^{(3)}(t)$. Convergence results of nonpolynomial Jacobi spectral collocation solution with $\alpha=\beta=\frac{1}{2}$, $\lambda=\frac{1}{3}$ and $\frac{1}{6}$, $\gamma=\frac{2}{3}$; $\lambda=\frac{1}{2}$ and $\frac{1}{4}$, $\gamma=\frac{1}{2}$; $\lambda=\frac{1}{3}$ and $\frac{1}{6}$, $\gamma=\frac{2}{3}$ are shown in Figs. 2-4, respectively.

From Figs. 2-4, we can find that exponential convergence occurred and the errors decay faster when $\lambda=1/3$, $\frac{1}{2}$, $1/3$ (left) than another cases when $\lambda=1/6$, $1/4$, $1/6$ (right), which verifies the conclusion of the optimal choice.

Example 5.3. Consider the following weakly singular FIE of the second kind with irrational exponent

$$x(t)-\int_0^1 |t-s|^{-\frac{1}{\sqrt{2}}}x(s)ds=f(t),$$

where $f(t)$ is chosen such that the exact solution $x(t)=1+t^{1/\sqrt{2}}+(1-t)^{1/\sqrt{2}}$.

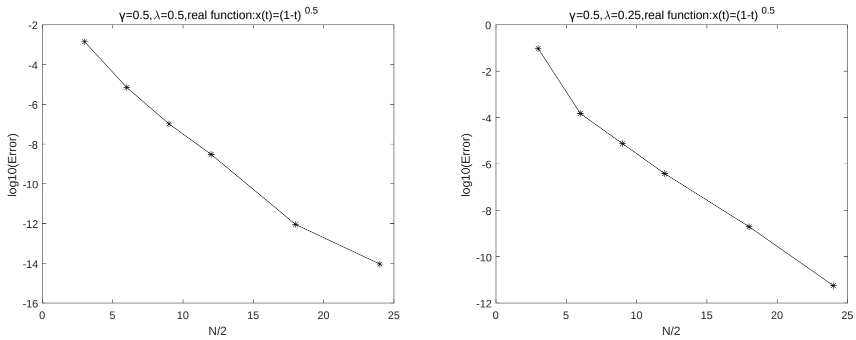

Figure 3: L^∞-norm errors for nonpolynomial Jacobi spectral collocation solutions for λ-polynomial degree N (DOF$=2N+2$) with $x^{(2)}(t)$.

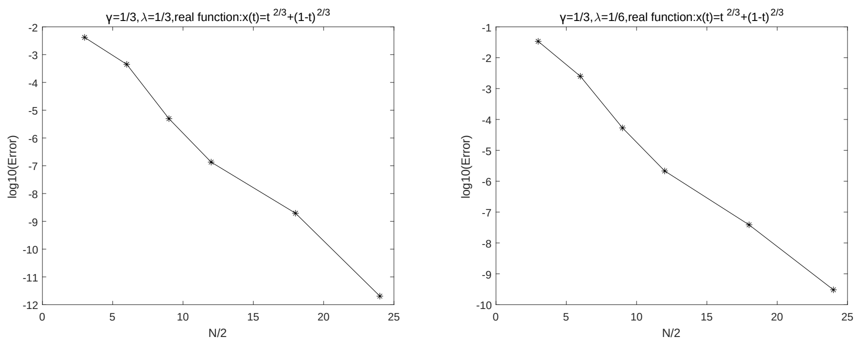

Figure 4: L^∞-norm errors for nonpolynomial Jacobi spectral collocation solutions for λ-polynomial degree N (DOF$=2N+2$) with $x^{(3)}(t)$.

The solution is weakly singular at the two endpoints. From Theorem 4.1, the optimal λ should be $1-1/\sqrt{2}$. The exponential convergence result of nonpolynomial Jacobi spectral collocation method is shown in Fig. 5.

Example 5.4. Consider the following two dimensional weakly singular FIE of the second kind

$$5u(x,y)-\int_0^1\int_0^1 |x-t|^{-\gamma_1}|y-s|^{-\gamma_2}u(s,t)dtds=g(x,y), \quad x,y\in(0,1),$$

where $g(x,y)$ is given by the exact solution

$$u(x,y)=x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\cdot y^{\frac{1}{2}}(1-y)^{\frac{1}{2}}.$$

In this example, the weak singularity occurs at the boundary. The convergence results of nonpolynomial Jacobi spectral collocation solution with $\alpha=\beta=-\frac{1}{2}$ are shown in Fig. 6. The exponential convergence rate is illustrated the theoretical result of the proposed method with $\lambda=\frac{1}{2}$ and $\frac{1}{4}$ for $\gamma_1=\gamma_2=\frac{1}{2}$. From Fig. 6, we can find that the error

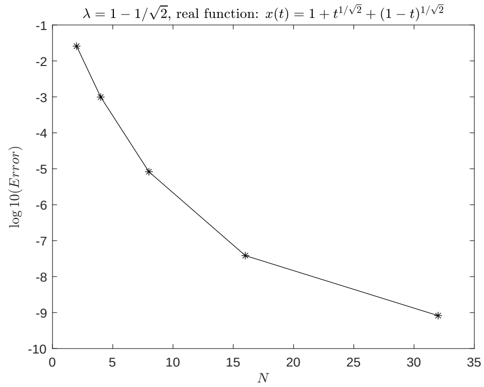

Figure 5: L^∞-norm errors for nonpolynomial Jacobi spectral collocation solutions with $\lambda = 1 - 1/\sqrt{2}$.

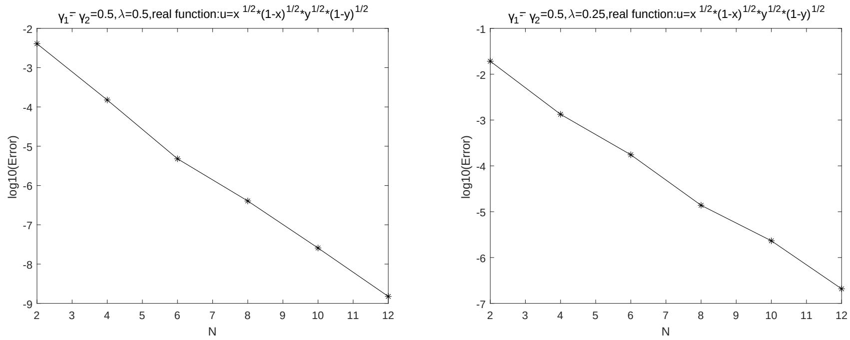

Figure 6: L^∞-norm errors for nonpolynomial Jacobi spectral collocation solutions with λ-polynomial degree N for two-dimensional FIE (DOF$=(2N+2)*(2N+2)$).

decays faster when $\lambda = \frac{1}{2}$ than the case when $\lambda = 1/4$, which verifies that $\lambda = \frac{1}{2}$ is the optimal choice.

6 Conclusions

In this paper, the nonpolynomial Jacobi spectral-collocation (NJSC) method for weakly singular FIEs of the second kind with low regularity exact solution is developed. For the FIE, we split the integral into two integrals and the FIE is rewritten as FIE system which include two equations. Then respectively using the variable changes and NJSC method to each equation, the mild singularities can be dealt with meanwhile. The similar idea can be applied to the two dimension case. The convergence analysis is given for the proposed method. We also give suggestions about the selection of the optimal λ for this efficient method. The numerical examples are given to verify the computation efficiency of the

NJSC method.

In the future, the following works will be done:

1. Log orthogonal functions (LOFs) and generalized log orthogonal functions (GLOFs) will be applied to solve (two dimensional) weakly singular Volterra integral equations of the second kind respectively.

2. A mapped Hermite Jacobi spectral collocation method with preconditioning techniques will be applied to solve (2-dimensional) weakly singular Fredholm integral equations of the second kind respectively.

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