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Error Analysis of a Pressure Penalty Scheme for the Reformulated Ericksen-Leslie System with Variable Density

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Abstract. Numerical approximation of the Ericksen-Leslie system with variable density is considered in this paper. The spherical constraint condition of the orientation field is preserved by using polar coordinates to reformulate the system. The equivalent new system is computationally cheaper because the vector function of the orientation field is replaced by a scalar function. An iteration penalty method is applied to construct a numerical scheme so that stability is improved. We first prove that the scheme is unique solvable and unconditionally stable in energy. Then we show that this scheme is of first-order convergence rate by rigorous error estimation. Finally, some numerical simulations are performed to illustrate the accuracy and effectiveness of the scheme.

AMS subject classifications: 65M12, 65M60

Key words: Variable density, constraint-preserving, Ericksen-Leslie, error analysis.

1 Introduction

In recent years, more and more scholars are interested in the theory of liquid crystals. Liquid crystals are materials that show the intermediate phase between solid and liquid. This implies that liquid crystals conjoin the characteristics of both solids and isotropic liquids. Nematic is the simplest phase of liquid crystals. In this case, molecules are provided with ordered orientation, but disordered in position configuration. In the 1960s, Ericksen [9] and Leslie [18] first introduced the Ericksen-Leslie system, which models the hydrodynamics of nematic liquid crystals. Under the influence of flow velocity and

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microscopic orientation configurations, this system describes the macroscopic temporal evolution of liquid crystal materials.

Lin [21] proposed a simplified version because the original Ericksen-Leslie system is too complicated. Major mathematical difficulties remain, although the simplified version ignores the Leslie tension. This system consists of a Navier-Stokes equation [29] coupled with the extra term $\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})$ and a harmonic map heat flow with the convection term $(\mathbf{u}\cdot\nabla)\mathbf{d}$ [22], read as:

$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla P-\mu\Delta\mathbf{u}+\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})=0, \tag{1.1a}$$

$$\mathbf{d}_t+(\mathbf{u}\cdot\nabla)\mathbf{d}-\Delta\mathbf{d}-|\nabla\mathbf{d}|^2\mathbf{d}=0, \tag{1.1b}$$

$$|\mathbf{d}|=1, \tag{1.1c}$$

$$\nabla\cdot\mathbf{u}=0, \tag{1.1d}$$

where $\mathbf{u}$, $\mathbf{d}$ and P are the fluid velocity, the mean orientation of the molecules and the fluid pressure, respectively. Coefficient μ of $\Delta\mathbf{u}$ represents the viscosity of the fluid. Some descriptions of operators in this system are given as follows. The gradient operator $\nabla\mathbf{u}=(\partial_j u_i)_{i,j}$; the Laplacian operator $\Delta\mathbf{u}=\sum_{i=1}^M\partial_{ii}\mathbf{u}$; the convective operator $(\mathbf{u}\cdot\nabla)\mathbf{d}=\sum_{i=1}^M u_i\partial_i\mathbf{d}$. Moreover, $|\mathbf{d}|$ is the Euclidean norm in $\mathbf{R}^M$. The term $\nabla\mathbf{d}\odot\nabla\mathbf{d}$ is a 2×2 matrix whose (i,j)-the entry is given by $(\nabla_i\mathbf{d})\cdot(\nabla_j\mathbf{d})$. Henceforth, we denote

$$\nabla\mathbf{d}\odot\nabla\mathbf{d}=(\nabla\mathbf{d})^T\nabla\mathbf{d},$$

where $(\nabla\mathbf{d})^T$ denotes the transpose of $\nabla\mathbf{d}$.

For the system (1.1), Lin et al. [26] proposed a C_0 finite element scheme for simulating the kinematic effects in liquid crystal dynamics. To obtain a flow equation without $\Delta\mathbf{d}$, they reformulated the flow equation by using the orientation field equation. In addition, they proved the discrete energy law. An and Su [3] investigated the time-dependent nematic liquid crystal flows by semi-implicit Galerkin method. They showed the temporal and the spatial error estimates. We refer the reader to [4, 13] and reference therein.

There are two main difficulties in studying the system (1.1). The spherical constraint condition $|\mathbf{d}|=1$ is difficult to implement at the discrete level. Specifically, we can not imply the spherical constraint at nodes by interpolation. Moreover, the extra term $\nabla\cdot((\nabla\mathbf{d})^T\nabla\mathbf{d})$ causes strong coupling. Therefore, a Ginzburg-Landau penalty method is proposed to overcome the difficulty of $|\mathbf{d}|=1$ [23]. By introducing a Ginzburg-Landau penalty function $\frac{1}{\epsilon^2}\mathbf{f}(\mathbf{d})$ to replace $|\nabla\mathbf{d}|^2\mathbf{d}$, the constraint $|\mathbf{d}|=1$ is relaxed. The general penalty version reads as follows:

$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla P=\mu\Delta\mathbf{u}-\nabla\cdot((\nabla\mathbf{d}^T)\nabla\mathbf{d}), \tag{1.2a}$$

$$\mathbf{d}_t+(\mathbf{u}\cdot\nabla)\mathbf{d}+\frac{1}{\epsilon^2}\mathbf{f}(\mathbf{d})-\Delta\mathbf{d}=0, \tag{1.2b}$$

$$\nabla\cdot\mathbf{u}=0, \tag{1.2c}$$

where $\mathbf{f}(\mathbf{d})$ is the penalty function, and $\epsilon>0$ is the penalty parameter. The penalty function is the derivative with respect to $\mathbf{d}$ of the function $F(\mathbf{d})$, in detail, $\mathbf{f}(\mathbf{d})=\nabla_{\mathbf{d}}F(\mathbf{d})$, where

$$F(\mathbf{d})=\begin{cases} \frac{1}{4}(|\mathbf{d}|^2-1)^2, & \text{if } |\mathbf{d}|\leq 1, \\ (|\mathbf{d}|^2-1)^2, & \text{if } |\mathbf{d}|>1. \end{cases}$$

Girault and Guillén-González [11] proposed a linear fully discrete mixed scheme for solving a penalized nematic liquid crystal model. They used C_0 finite elements in space and semi-implicit Euler scheme in time to obtain unconditional stability in energy. Furthermore, the first-order optimal error estimates are proved. By using a saddle-point formulation, Badia et al. [5] proposed a finite element scheme for numerical approximation of the nematic liquid crystal flows. They introduced a Lagrange multiplier that enforces the sphere condition, so that the limit problem (without penalty) and the penalized problem (using a Ginzburg-Landau penalty function) can be considered in a unified way. Some other research on penalty Ericksen-Leslie equations can be found in [7,16,24,36].

Although there is a great deal of literatures about penalty Ericksen-Leslie system, it is still an open and challenging problem that whether weak solutions $(\mathbf{u}_\epsilon,\mathbf{d}_\epsilon)$ of the system (1.2) weakly converge to that of the system (1.1) as $\epsilon\to 0$ [25]. Therefore, it makes sense to seek another method to study the system (1.1). In the 2D case, a new method to deal with the spherical constraint condition $|\mathbf{d}|=1$ is rewriting the harmonic map heat flow by polar coordinates, i.e., denoting $\mathbf{d}(\mathbf{x},t)=(d_1,d_2)^T$ as

$$\mathbf{d}(\mathbf{x},t)=(\cos\theta(\mathbf{x},t),\sin\theta(\mathbf{x},t))^T,$$

where $\theta(\mathbf{x},t)=\arg\mathbf{d}(\mathbf{x},t)$ with arg being the argument from the polar coordinates. Compared with the Ginzburg-Landau penalty function, this method preserved the spherical constraint condition $|\mathbf{d}|=1$. Similarly, by using spherical coordinates, one can denotes $\mathbf{d}(\mathbf{x},t)=(d_1,d_2,d_3)^T$ as

$$\mathbf{d}(\mathbf{x},t)=(\cos\theta,\sin\theta\cos\psi,\sin\theta\sin\psi)^T,$$

in the 3D case [17]. In this situation, the orientation field equation transformed into two equations:

$$\begin{aligned} &\theta_t+\mathbf{u}\cdot\nabla\theta=\Delta\theta-\sin\theta\cos\theta|\nabla\psi|^2, \\ &\psi_t+\mathbf{u}\cdot\nabla\psi=\Delta\psi+2\cot\theta\nabla\theta\cdot\nabla\psi, \end{aligned}$$

where the definition of ψ is the same as that of θ.

Gong et al. [14] studied a general Ericksen-Leslie system by using the above method in two dimensions. Nonlinear stress terms and transport terms induce some difficulties in the analysis. To overcome these problems, they introduced an elliptic operator. Then they proved the existence of global strong solutions. Bao et al. [6] rewrote the simplified Ericksen-Leslie system by using the above method and proposed an energy stable numerical scheme for the new system. By proving a discrete maximum principle of the scheme, they ensured equivalence between the new and original system. Moreover, they showed that the scheme is uniquely solvable and satisfies a discrete energy law.

Recently, many mathematicians have been absorbed in investigating dynamic systems with variable density. An [1] proposed a fractional-step scheme for the numerical solution of the incompressible Navier-Stokes equations with variable density. He proved the energy stablility and the first-order temporal error estimates. Zhu et al. [37] investigated a phase-field moving contact line model with variable densities and viscosities. This model comprises Cahn-Hilliard equation and Navier-Stokes equation. In addition, a scalar auxiliary variable was used to transform the system into an equivalent form, which allowed the double well potential to be treated semi-explicitly. About Cahn-Hilliard equation, we refer the reader to [30–35] and references therein. To the best of our knowledge, there is little research on the numerical method of the Ericksen-Leslie system with variable density.

Inspired by the above research, we will consider the simplified incompressible Ericksen-Leslie system with variable density:

$$\rho_t+\nabla\cdot(\rho\mathbf{u})=0, \tag{1.3a}$$

$$\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})+\nabla P-\mu\Delta\mathbf{u}+\nabla\cdot((\nabla\mathbf{d})^T\nabla\mathbf{d})=0, \tag{1.3b}$$

$$\mathbf{d}_t+(\mathbf{u}\cdot\nabla)\mathbf{d}-\Delta\mathbf{d}-|\nabla\mathbf{d}|^2\mathbf{d}=0, \tag{1.3c}$$

$$|\mathbf{d}|=1, \tag{1.3d}$$

$$\nabla\cdot\mathbf{u}=0, \tag{1.3e}$$

in $[0,T]\times\Omega$, where ρ is density of the liquid crystals. We consider the boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0, \quad \partial_{\mathbf{n}}\mathbf{d}|_{\partial\Omega}=0,$$

and initial data

$$\rho(x,0)=\rho_0, \quad \mathbf{u}(x,0)=\mathbf{u}_0, \quad \mathbf{d}(x,0)=\mathbf{d}_0.$$

This system satisfies the following energy law [27]:

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega(\rho|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)dx+\int_\Omega(\mu|\nabla\mathbf{u}|^2+|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}|^2)dx\right)=0.$$

Furthermore, Liu [27] established the global well-posedness of strong solutions in the vacuum cases under some assumptions. Fan et al. [10] established the existence and uniqueness of strong solutions with vacuum in a bounded smooth domain.

This paper is organized as follows. In Section 2, we give some notations and assumptions. We rewrite the system (1.3) by polar coordinates as aforementioned. In Section 3, we construct a numerical scheme for the new system and derive the unconditional stability in energy. In Section 4, we prove that this scheme is of first-order convergence rate $\mathcal{O}(\tau)$. Some numerical experiments are given in Section 5.

2 Preliminaries

We first introduce some notations. Let $\Omega \in \mathbf{R}^2$ be a bounded open set with boundary $\partial\Omega$. We denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$, and by $\|u\| = (u,u)^{\frac{1}{2}}$ its norm. $W^{k,p}(\Omega)$ is the classical Sobolev space. It becomes the Hilbert space $H^k(\Omega)$ when $p=2$. Let $\mathcal{D}(\Omega)$ be the space of infinitely times differentiable functions on Ω with compact support. The closure of $\mathcal{D}(\Omega)$ in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$. All these definitions and notations carry over to two-dimensional vector function spaces. Throughout this paper, the symbol C denote some positive constants which are independent of the time step size τ. Thanks to [6], we can use polar coordinate to rewrite the system (1.3), i.e.,

$$\mathbf{d}(\mathbf{x},t) = (\cos\theta(\mathbf{x},t), \sin\theta(\mathbf{x},t))^T.$$

For the velocity field equation, a direct calculation shows that

$$\begin{aligned}
&\nabla \mathbf{d} = \begin{bmatrix} d_{1\theta}\theta_x & d_{1\theta}\theta_y \\ d_{2\theta}\theta_x & d_{2\theta}\theta_y \end{bmatrix} = \begin{bmatrix} -\theta_x \sin\theta & -\theta_y \sin\theta \\ \theta_x \cos\theta & \theta_y \cos\theta \end{bmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} (\theta_x, \theta_y), \\
&\Delta d_1 = (d_{1\theta}\theta_x)_x + (d_{1\theta}\theta_y)_y = -|\nabla\theta|^2 \cos\theta - \Delta\theta \sin\theta, \\
&\Delta d_2 = (d_{2\theta}\theta_x)_x + (d_{2\theta}\theta_y)_y = -|\nabla\theta|^2 \sin\theta + \Delta\theta \cos\theta, \\
&\Delta \mathbf{d} = (\Delta d_1, \Delta d_2)^T = \Delta\theta(-\sin\theta, \cos\theta)^T - (\theta_x^2 + \theta_y^2)(\cos\theta, \sin\theta)^T.
\end{aligned}$$

Then

$$\nabla \cdot ((\nabla \mathbf{d})^T \nabla \mathbf{d}) = (\nabla \mathbf{d})^T \Delta \mathbf{d} + \frac{1}{2} \nabla (|\nabla \mathbf{d}|^2) = \Delta\theta \nabla\theta + \frac{1}{2} \nabla (|\nabla\theta|^2).$$

Similarly, with regard to the orientation field equation, we have

$$\mathbf{d}_t = (d_{1\theta}\theta_t, d_{2\theta}\theta_t)^T = \theta_t(-\sin\theta, \cos\theta)^T,$$

and

$$|\nabla \mathbf{d}|^2 = \theta_x^2 + \theta_y^2.$$

Thus,

$$\begin{aligned}
&\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \\
=&\Delta\theta(-\sin\theta, \cos\theta)^T - (\theta_x^2 + \theta_x^2)(\cos\theta, \sin\theta)^T + (\theta_x^2 + \theta_x^2)(\cos\theta, \sin\theta)^T \\
=&\Delta\theta(-\sin\theta, \cos\theta)^T.
\end{aligned}$$

Therefor, the third equation of (1.3) can be rewritten as:

$$\theta_t(-\sin\theta, \cos\theta) + \mathbf{u} \cdot \nabla\theta(-\sin\theta, \cos\theta) = \Delta\theta(-\sin\theta, \cos\theta).$$

Then, the system (1.3) can be read as:

$$\rho_t+\nabla\cdot(\rho\mathbf{u})=0, \tag{2.1a}$$
$$\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})+\nabla\tilde{P}-\mu\Delta\mathbf{u}+\Delta\theta\nabla\theta=0, \tag{2.1b}$$
$$\theta_t+\mathbf{u}\cdot\nabla\theta-\Delta\theta=0, \tag{2.1c}$$
$$\nabla\cdot\mathbf{u}=0, \tag{2.1d}$$

in $[0,T]\times\Omega$, with boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0, \quad \nabla\theta\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{2.2}$$

and initial conditions

$$\rho(x,0)=\rho_0, \quad \mathbf{u}(x,0)=\mathbf{u}_0, \quad \theta(x,0)=\theta_0, \tag{2.3}$$

where $\tilde{P}=P+\frac{1}{2}|\nabla\theta|^2$.

Remark 2.1. The singularities of orientation field $\mathbf{d}$ are defined as the points where $|\mathbf{d}|=0$. Thus, the system (2.1) can prevent the formation of singularities since $|\mathbf{d}|\equiv 1$.

3 Unconditional energy stable scheme

An unconditional energy stable discrete scheme of the system (2.1)-(2.3) will be given in this section. Since $\nabla\cdot\mathbf{u}=0$, we can rewrite (2.1a) and (2.1b) as follows:

$$\rho_t+\nabla\rho\cdot\mathbf{u}+\frac{1}{2}\rho(\nabla\cdot\mathbf{u})=0,$$
$$\rho\mathbf{u}_t+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{4}\rho(\nabla\cdot\mathbf{u})\mathbf{u}+\nabla\tilde{P}-\mu\Delta\mathbf{u}+\Delta\theta\nabla\theta=0.$$

We denote the time step $\tau=T/N$ and the discrete time $t_n=n\tau$ for $1\le n\le N$. Because $\frac{1}{2}\|\sqrt{\rho^n}\mathbf{u}^n\|^2$ is the kinetic energy of the flow, it is more appropriate to establish bounds based on $\|\sqrt{\rho^n}\mathbf{u}^n\|^2$ than on velocity itself. For simplicity, we denote $\sigma^n=\sqrt{\rho^n}$ for all $1\le n\le N$ and $\sigma_0=\sqrt{\rho_0}$.

Given the initial conditions

$$\rho^0=\rho_0, \quad \mathbf{u}^0=\mathbf{u}_0, \quad \tilde{P}^0=\tilde{P}_0 \quad \text{and} \quad \theta^0=\theta_0,$$

having computed for ρ^n, $\mathbf{u}^n$, $\tilde{P}^n$, and θ^n, we compute ρ^{n+1}, $\mathbf{u}^{n+1}$, $\tilde{P}^{n+1}$ and θ^{n+1} by

$$\frac{\rho^{n+1}-\rho^n}{\tau}+\nabla\rho^{n+1}\cdot\mathbf{u}^n+\frac{1}{2}\rho^{n+1}(\nabla\cdot\mathbf{u}^n)=0, \tag{3.1a}$$
$$\begin{aligned}\rho^n\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau}&+\rho^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1}+\frac{1}{4}\rho^{n+1}(\nabla\cdot\mathbf{u}^n)\mathbf{u}^{n+1}\\ &+\nabla\tilde{P}^{n+1}-\mu\Delta\mathbf{u}^{n+1}+\Delta\theta^{n+1}\nabla\theta^n=0,\end{aligned} \tag{3.1b}$$
$$\frac{\theta^{n+1}-\theta^n}{\tau}+\mathbf{u}^{n+1}\cdot\nabla\theta^n-\Delta\theta^{n+1}=0, \tag{3.1c}$$
$$\varepsilon(\tilde{P}^{n+1}-\tilde{P}^n)+\nabla\cdot\mathbf{u}^{n+1}=0, \tag{3.1d}$$

where $\varepsilon>0$ is the penalty parameter [2]. It is clear that (3.1a) is uniquely solvable. We next prove the existence and uniqueness of the solution to (3.1b)-(3.1d). Since (3.1b)-(3.1d) having the same number of unknowns as equations, uniqueness implies existence. Let $\hat{\mathbf{u}}^{n+1}$, $\hat{\theta}^{n+1}$ and $\hat{P}^{n+1}$ denote the difference between two possible solutions, we can see that they satisfy

$$\begin{aligned}&\frac{\rho^n}{\tau}\hat{\mathbf{u}}^{n+1}+\rho^{n+1}\mathbf{u}^n\cdot\nabla\hat{\mathbf{u}}^{n+1}+\frac{1}{4}\rho^{n+1}(\nabla\cdot\mathbf{u}^n)\hat{\mathbf{u}}^{n+1}\\&\quad+\nabla\hat{P}^{n+1}-\mu\Delta\hat{\mathbf{u}}^{n+1}+\Delta\hat{\theta}^{n+1}\nabla\theta^n=0,\end{aligned}\tag{3.2a}$$

$$\frac{1}{\tau}\hat{\theta}^{n+1}+\hat{\mathbf{u}}^{n+1}\cdot\nabla\theta^n-\Delta\hat{\theta}^{n+1}=0,\tag{3.2b}$$

$$\varepsilon\hat{P}^{n+1}+\nabla\cdot\hat{\mathbf{u}}^{n+1}=0.\tag{3.2c}$$

Testing (3.2a)-(3.2c) by $\hat{\mathbf{u}}^{n+1}$, $-\Delta\hat{\theta}^{n+1}$ and $\hat{P}^{n+1}$ respectively, multiplying (3.1a) by $|\hat{\mathbf{u}}^{n+1}|^2$, we have

$$\frac{1}{\tau}(\|\sigma^{n+1}\hat{\mathbf{u}}^{n+1}\|^2+\|\nabla\hat{\theta}^{n+1}\|^2)+\varepsilon\|\hat{P}^{n+1}\|^2+\mu\|\nabla\hat{\mathbf{u}}^{n+1}\|^2+\|\Delta\hat{\theta}^{n+1}\|^2=0,$$

which implies that $\hat{\mathbf{u}}^{n+1}=0$ and $\hat{P}^{n+1}=0$. Testing (3.2b) by $\hat{\theta}^{n+1}$, since $\hat{\mathbf{u}}^{n+1}=0$, we derive

$$\frac{1}{\tau}\|\hat{\theta}^{n+1}\|^2+\|\nabla\hat{\theta}^{n+1}\|^2=0,$$

which implies that $\hat{\theta}^{n+1}=0$. Thus, we have proved uniqueness and existence of (3.1b)-(3.1d).

Theorem 3.1. *For any $\tau>0$, the system* (3.1a)-(3.1d) *satisfy the following energy estimates:*

$$\|\rho^{m+1}\|^2+\sum_{n=0}^{m}\|\rho^{n+1}-\rho^n\|^2=\|\rho^0\|^2,\tag{3.3}$$

and

$$\begin{aligned}&\|\sigma^{m+1}\mathbf{u}^{m+1}\|^2+\varepsilon\tau\|\tilde{P}^m\|^2+\|\nabla\theta^{m+1}\|^2+2\tau\sum_{n=0}^{m}(\mu\|\nabla\mathbf{u}^{n+1}\|^2+\|\Delta\theta^{n+1}\|^2)\\&\quad+\sum_{n=0}^{m}(\|\sigma^n(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2+\varepsilon\tau\|\tilde{P}^{n+1}-\tilde{P}^n\|^2+\|\nabla\theta^{n+1}-\nabla\theta^n\|^2)\\&=\|\sigma^0\mathbf{u}^0\|^2+\varepsilon\tau\|\tilde{P}^0\|^2+\|\nabla\theta^0\|^2.\end{aligned}\tag{3.4}$$

Proof. Testing (3.1a) by ρ^{n+1}, we deduce that

$$\begin{aligned}&\frac{1}{2\tau}(\|\rho^{n+1}\|^2-\|\rho^n\|^2+\|\rho^{n+1}-\rho^n\|^2)+(\nabla\rho^{n+1}\cdot\mathbf{u}^n,\rho^{n+1})\\&\quad+\frac{1}{2}(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),\rho^{n+1})=0.\end{aligned}$$

According to the boundary condition of velocity, we note that

$$\begin{aligned}&(\nabla\rho^{n+1}\cdot\mathbf{u}^n,\rho^{n+1})+\frac{1}{2}(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),\rho^{n+1})\\=&\frac{1}{2}\int_\Omega\nabla\cdot(|\rho^{n+1}|^2\cdot\mathbf{u}^n)dx\\=&\frac{1}{2}\int_{\partial\Omega}|\rho^{n+1}|^2\cdot\mathbf{u}^n\cdot\mathbf{n}dx=0,\end{aligned}$$

which implies

$$\|\rho^{n+1}\|^2-\|\rho^n\|^2+\|\rho^{n+1}-\rho^n\|^2=0.$$

For any integer $m\in(0,N]$, summing from 0 to m, we get

$$\|\rho^{m+1}\|^2+\sum_{n=0}^m\|\rho^{n+1}-\rho^n\|^2=\|\rho^0\|^2.$$

Taking the inner product of (3.1a) with $\frac{1}{2}|\mathbf{u}^{n+1}|^2$, (3.1b) with $\mathbf{u}^{n+1}$ and (3.1c) with $-\Delta\theta^{n+1}$, we have

$$\begin{aligned}&\frac{1}{2\tau}(\|\sigma^{n+1}\mathbf{u}^{n+1}\|^2-\|\sigma^n\mathbf{u}^{n+1}\|^2)+\frac{1}{2}(\nabla\rho^{n+1}\cdot\mathbf{u}^n,|\mathbf{u}^{n+1}|^2)\\&\quad+\frac{1}{4}(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),|\mathbf{u}^{n+1}|^2)=0, &(3.5a)\\&\frac{1}{2\tau}(\|\sigma^n\mathbf{u}^{n+1}\|^2-\|\sigma^n\mathbf{u}^n\|^2+\|\sigma^n(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2)+\mu\|\nabla\mathbf{u}^{n+1}\|^2\\&\quad+(\rho^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1},\mathbf{u}^{n+1})+\frac{1}{4}(\rho^{n+1}(\nabla\cdot\mathbf{u}^n)\mathbf{u}^{n+1},\mathbf{u}^{n+1})\\&\quad+(\nabla\tilde{P}^{n+1},\mathbf{u}^{n+1})+(\Delta\theta^{n+1}\nabla\theta^n,\mathbf{u}^{n+1})=0, &(3.5b)\\&\frac{1}{2\tau}(\|\nabla\theta^{n+1}\|^2-\|\nabla\theta^n\|^2+\|\nabla\theta^{n+1}-\nabla\theta^n\|^2)\\&\quad-(\mathbf{u}^{n+1}\cdot\nabla\theta^n,\Delta\theta^{n+1})+\|\Delta\theta^{n+1}\|^2=0. &(3.5c)\end{aligned}$$

From (3.1d), we infer that

$$2\tau(\nabla\tilde{P}^{n+1},\mathbf{u}^{n+1})=\varepsilon\tau\|\tilde{P}^{n+1}\|^2-\varepsilon\tau\|\tilde{P}^n\|^2+\varepsilon\tau\|\tilde{P}^{n+1}-\tilde{P}^n\|^2. \quad (3.6)$$

Note that

$$\begin{aligned}&(\rho^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1},\mathbf{u}^{n+1})+\frac{1}{4}(\rho^{n+1}(\nabla\cdot\mathbf{u}^n)\mathbf{u}^{n+1},\mathbf{u}^{n+1})\\&\quad+\frac{1}{2}(\nabla\rho^{n+1}\cdot\mathbf{u}^n,|\mathbf{u}^{n+1}|^2)+\frac{1}{4}(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),|\mathbf{u}^{n+1}|^2)\\=&\frac{1}{2}\int_\Omega\nabla\cdot(\rho^{n+1}\mathbf{u}^n\cdot|\mathbf{u}^{n+1}|^2)dx\\=&\frac{1}{2}\int_{\partial\Omega}\rho^{n+1}\mathbf{u}^n\cdot|\mathbf{u}^{n+1}|^2\cdot\mathbf{n}dx=0.\end{aligned}$$

Combining with (3.5a)-(3.6), we obtain that

$$
\begin{aligned}
&\|\sigma^{n+1}\mathbf{u}^{n+1}\|^2+\varepsilon\tau\|\tilde{P}^{n+1}\|^2+\|\nabla\theta^{n+1}\|^2+\|\sigma^n(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2\\
&\quad+\varepsilon\tau\|\tilde{P}^{n+1}-\tilde{P}^n\|^2+\|\nabla\theta^{n+1}-\nabla\theta^n\|^2+2\tau(\mu\|\nabla\mathbf{u}^{n+1}\|^2+\|\Delta\theta^{n+1}\|^2)\\
=&\|\sigma^n\mathbf{u}^n\|^2+\varepsilon\tau\|\tilde{P}^n\|^2+\|\nabla\theta^n\|^2.
\end{aligned}
$$

For any integer $m\in(0,N]$, summing from 0 to m, we can see that

$$
\begin{aligned}
&\|\sigma^{m+1}\mathbf{u}^{m+1}\|^2+\varepsilon\tau\|\tilde{P}^m\|^2+\|\nabla\theta^{m+1}\|^2+2\tau\sum_{n=0}^{m}(\mu\|\nabla\mathbf{u}^{n+1}\|^2+\|\Delta\theta^{n+1}\|^2)\\
&\quad+\sum_{n=0}^{m}(\|\sigma^n(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2+\varepsilon\tau\|\tilde{P}^{n+1}-\tilde{P}^n\|^2+\|\nabla\theta^{n+1}-\nabla\theta^n\|^2)\\
=&\|\sigma^0\mathbf{u}^0\|^2+\varepsilon\tau\|\tilde{P}^0\|^2+\|\nabla\theta^0\|^2.
\end{aligned}
$$

This completes the proof. □

We will use the finite element method for the spatial discretization. Let $\mathcal{T}_h$ be a family of quasi-uniform triangular partition of $\overline{\Omega}$, the ordered triangles are denoted by K_i, $(i=1,\dots,B)$. Let $h_i=diam(K_i)$, denote by $h=\max\{h_1,h_2,\cdots,h_B\}$ the mesh size. Let us denote $\mathbf{V}_h\subset\mathbf{H}_0^1(\Omega)$, $M_h\subset L_0^2(\Omega)$ and $W_h,G_h\subset H^1(\Omega)$ be the finite-dimensional subspaces, where $L_0^2(\Omega)=\{q\in L^2(\Omega),\ \int_\Omega q dx=0\}$. More specifically,

$$
\begin{aligned}
&W_h=\{w_h\in\mathcal{C}(\overline{\Omega})\,|\,w_h\in P_2(K),\ \forall K\in\mathcal{T}_h\},\\
&\mathbf{V}_h=\{\mathbf{v}_h\in\mathcal{C}(\overline{\Omega})^2\,|\,\mathbf{v}_h\in P_2(K)^2,\ \forall K\in\mathcal{T}_h\},\\
&G_h=\{g_h\in\mathcal{C}(\overline{\Omega})\,|\,g_h\in P_1(K),\ \forall K\in\mathcal{T}_h\},\\
&M_h=\{q_h\in\mathcal{C}(\overline{\Omega})\,|\,q_h\in P_1(K),\ \forall K\in\mathcal{T}_h\}.
\end{aligned}
$$

Then, the fully discretized scheme is described as follows: find $\rho_h^{n+1}\in W_h$, $\mathbf{u}_h^{n+1}\in\mathbf{V}_h$, $\tilde{P}_h^{n+1}\in M_h$ and $\theta_h^{n+1}\in G_h$, such that

$$
\begin{aligned}
&\frac{1}{\tau}(\rho_h^{n+1}-\rho_h^n,w_h)+(\nabla\rho_h^{n+1}\cdot\mathbf{u}_h^n,w_h)+\frac{1}{2}(\rho_h^{n+1}(\nabla\cdot\mathbf{u}_h^n),w_h)=0, && \forall w_h\in W_h,\\
&\frac{1}{\tau}(\rho_h^n(\mathbf{u}_h^{n+1}-\mathbf{u}_h^n),\mathbf{v}_h)+(\rho_h^{n+1}\mathbf{u}_h^n\cdot\nabla\mathbf{u}_h^{n+1},\mathbf{v}_h)+\frac{1}{4}(\rho_h^{n+1}(\nabla\cdot\mathbf{u}_h^n)\mathbf{u}_h^{n+1},\mathbf{v}_h)\\
&\quad-(\tilde{P}_h^{n+1},\nabla\cdot\mathbf{v}_h)+\mu(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h)+(\Delta\theta_h^{n+1}\nabla\theta_h^n,\mathbf{v}_h)=0, && \forall\mathbf{v}_h\in\mathbf{V}_h,\\
&\frac{1}{\tau}(\theta_h^{n+1}-\theta_h^n,g_h)+(\mathbf{u}_h^{n+1}\cdot\nabla\theta_h^n,g_h)+(\nabla\theta_h^{n+1},\nabla g_h)=0, && \forall g_h\in G_h,\\
&\varepsilon(\tilde{P}_h^{n+1}-\tilde{P}_h^n,q_h)+(\nabla\cdot\mathbf{u}_h^{n+1},q_h)=0, && \forall q_h\in M_h.
\end{aligned}
$$

It is clear that the energy estimates (3.3) and (3.4) still hold for the full discrete scheme.

4 Temporal error analysis

This section is devoted to the temporal error analysis for the scheme (3.1a)-(3.1d). We show that this scheme is of first-order convergence rate $\mathcal{O}(\tau)$. To this end, we need the following assumptions of density:

$$\begin{cases} \{\rho^n\}_{n=0,\cdots,N} \text{ is uniformly bounded in } L^\infty \\ \quad \text{for all } n=0,\cdots,N, \text{ there holds } \rho^n \geq \chi \text{ a.e. in } \Omega, \end{cases} \tag{4.1}$$

where χ is a number in $(0,\rho_0^{\min}]$. For more detailed discusses of rationality of (4.1), we refer to Remark 2.4 in [15], Remark 4.1 and Remark 4.2 in [1].

Furthermore, we make the following regularity assumptions on the exact solution to obtain the convergence rates of the scheme:

$$\rho \in H^2(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,\infty}(\Omega)), \tag{4.2a}$$
$$\mathbf{u} \in \mathbf{H}^2(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{H}_0^1 \cap \mathbf{H}^2(\Omega)), \tag{4.2b}$$
$$\tilde{P} \in L^\infty(0,T;L_0^2(\Omega) \cap H^1(\Omega)), \quad \tilde{P}_t, \tilde{P}_{tt} \in C([0,T];L_0^2(\Omega)), \tag{4.2c}$$
$$\theta \in H^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1 \cap H^2(\Omega)). \tag{4.2d}$$

We denote

$$e_\rho^n = \rho(t_n) - \rho^n, \quad e_u^n = \mathbf{u}(t_n) - \mathbf{u}^n, \quad \xi^n = \tilde{P}(t_n) - \tilde{P}^n, \quad e_\theta^n = \theta(t_n) - \theta^n.$$

Taking $t = t_{n+1}$ in (2.1) and subtracting from (3.1a)-(3.1d), since $\nabla \cdot \mathbf{u}(t_n) = 0$, we get

$$\begin{aligned} \frac{e_\rho^{n+1} - e_\rho^n}{\tau} =& -\nabla \rho(t_{n+1}) \cdot \mathbf{u}(t_{n+1}) + \nabla \rho^{n+1} \cdot \mathbf{u}^n \\ & - \frac{1}{2}\rho(t_{n+1})(\nabla \cdot \mathbf{u}(t_n)) + \frac{1}{2}\rho^{n+1}(\nabla \cdot \mathbf{u}^n) + R_\rho^{n+1}, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} \rho^n \frac{e_u^{n+1} - e_u^n}{\tau} =& -\rho(t_{n+1})\mathbf{u}(t_{n+1}) \cdot \nabla \mathbf{u}(t_{n+1}) + \rho^{n+1}\mathbf{u}^n \cdot \nabla \mathbf{u}^{n+1} \\ & - \frac{1}{4}\rho(t_{n+1})(\nabla \cdot \mathbf{u}(t_n))\mathbf{u}(t_{n+1}) + \frac{1}{4}\rho^{n+1}(\nabla \cdot \mathbf{u}^n)\mathbf{u}^{n+1} \\ & - \nabla \xi^{n+1} + \mu \Delta e_u^{n+1} - \Delta\theta(t_{n+1})\nabla\theta(t_{n+1}) + \Delta\theta^{n+1}\nabla\theta^n + R_u^{n+1}, \end{aligned} \tag{4.3b}$$

$$\varepsilon(\xi^{n+1} - \xi^n) = -\nabla \cdot e_u^{n+1} + \varepsilon(\tilde{P}(t_{n+1}) - \tilde{P}(t_n)), \tag{4.3c}$$

$$\frac{e_\theta^{n+1} - e_\theta^n}{\tau} = \Delta e_\theta^{n+1} - \mathbf{u}(t_{n+1}) \cdot \nabla\theta(t_{n+1}) + \mathbf{u}^{n+1} \cdot \nabla\theta^n + R_\theta^{n+1}, \tag{4.3d}$$

where

$$
\begin{aligned}
R_\rho^{n+1} &= \frac{\rho(t_{n+1})-\rho(t_n)}{\tau} - \rho_t(t_{n+1}),\\
R_u^{n+1} &= \rho^n \frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\tau} - \rho(t_{n+1})\mathbf{u}_t(t_{n+1}),\\
R_\theta^{n+1} &= \frac{\theta(t_{n+1})-\theta(t_n)}{\tau} - \theta_t(t_{n+1}).
\end{aligned}
$$

Lemma 4.1. *Assuming that the solution to* (2.1) *satisfies* (4.2) *and the assumption* (4.1) *hold. Then, we have*

$$
\begin{aligned}
&\|R_\rho^{n+1}\|^2 \le C\tau \int_{t_n}^{t_{n+1}} \|\rho_{tt}(t)\|^2 dt \le C\tau^2,\\
&\|R_u^{n+1}\|^2 \le C\tau \int_{t_n}^{t_{n+1}} (\|\mathbf{u}_{tt}(t)\|^2 + \|\rho_t(t)\|^2) dt + C_1\|e_\rho^n\|^2 \le C\tau^2 + C_1\|e_\rho^n\|^2,\\
&\|R_\theta^{n+1}\|^2 \le C\tau \int_{t_n}^{t_{n+1}} \|\theta_{tt}(t)\|^2 dt \le C\tau^2.
\end{aligned}
$$

Proof. Rewrite R_u^{n+1} as following

$$
R_u^{n+1} = \rho^n \left(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\tau} - \mathbf{u}_t(t_{n+1}) \right) - \left(e_\rho^n + \int_{t_n}^{t_{n+1}} \rho_t(t) dt \right) \mathbf{u}_t(t_{n+1}).
$$

According to the integral residual of the Taylor formula, we obtian

$$
R_u^{n+1} = \frac{\rho^n}{\tau} \int_{t_n}^{t_{n+1}} (t-t_n)\mathbf{u}_{tt}(t) dt - \left(e_\rho^n + \int_{t_n}^{t_{n+1}} \rho_t(t) dt \right) \mathbf{u}_t(t_{n+1}),
$$

which implies that

$$
\begin{aligned}
\|R_u^{n+1}\|^2 \le& 2\left\| \frac{\rho^n}{\tau} \int_{t_n}^{t_{n+1}} (t-t_n)\mathbf{u}_{tt}(t) dt \right\|^2 + 2\|e_\rho^n \mathbf{u}_t(t_{n+1})\|^2\\
&+ 2\left\| \left(\int_{t_n}^{t_{n+1}} \rho_t(t) dt \right) \mathbf{u}_t(t_{n+1}) \right\|^2\\
:=& K_1 + K_2 + K_3.
\end{aligned}
$$

From (4.1) and (4.2), using Hölder inequality, we can deduce that

$$
\begin{aligned}
K_1 \le& \frac{2}{\tau^2} \|\rho^n\|_{L^\infty}^2 \cdot \left\| \int_{t_n}^{t_{n+1}} (t-t_n)\mathbf{u}_{tt}(t) dt \right\|^2\\
\le& \frac{C}{\tau^2} \left\| \left(\int_{t_n}^{t_{n+1}} (t-t_n)^2 dt \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \Big(\mathbf{u}_{tt}(t)\Big)^2 dt \right)^{\frac{1}{2}} \right\|^2\\
\le& C\tau \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|^2 dt \le C\tau^2,
\end{aligned}
$$

$$
\begin{aligned}
K_2 \leq & 2\|e_\rho^n\|^2 \cdot \|\mathbf{u}(t_{n+1})\|_{L^\infty}^2 \leq C_1 \|e_\rho^n\|^2, \\
K_3 \leq & 2\left\| \int_{t_n}^{t_{n+1}} \rho_t(t) dt \right\|^2 \cdot \|\mathbf{u}(t_{n+1})\|_{L^\infty}^2 \\
\leq & C \left\| \left(\int_{t_n}^{t_{n+1}} 1^2 dt \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \rho_t^2(t) dt \right)^{\frac{1}{2}} \right\|^2 \\
\leq & C\tau \int_{t_n}^{t_{n+1}} \|\rho_t\|^2 dt \leq C\tau^2.
\end{aligned}
$$

Similarly, we can proof the inequalities of $\|R_\rho^{n+1}\|^2$ and $\|R_\theta^{n+1}\|^2$. □

Lemma 4.2 (Gronwall's inequality). *Let a_k, b_k, c_k and γ_k, for integers $k \geq 0$, be the nonnegative numbers such that*

$$
a_n + \tau \sum_{k=0}^{n} b_k \leq \tau \sum_{k=0}^{n} \gamma_k a_k + \tau \sum_{k=0}^{n} c_k + B \quad \textit{for} \ \ n \geq 0. \tag{4.4}
$$

Suppose that $\tau\gamma_k < 1$ for all k, and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then

$$
a_n + \tau \sum_{k=0}^{n} b_k \leq \exp\left(\tau \sum_{k=0}^{n} \gamma_k \sigma_k \right) \left(\tau \sum_{k=0}^{n} c_k + B \right) \quad \textit{for} \ \ n \geq 0. \tag{4.5}
$$

Remark 4.1. If the term $\tau \sum_{k=0}^{n} \gamma_k a_k$ in (4.4) extends only up to $n-1$, then the estimate (4.5) holds for all $0 < \tau < 1$ with $\sigma_k = 1$.

Theorem 4.1. *Assuming that the solutions to* (2.1) *satisfies* (4.2) *and the assumption* (4.1) *hold, then for sufficiently small τ, there are the following error estimates*

$$
\|e_\rho^N\|^2 + \sum_{n=0}^{N-1} \|e_\rho^{n+1} - e_\rho^n\|^2 \leq C\tau^2 + \chi^{-1} \tau \sum_{n=0}^{N-1} \|\sigma^n e_u^n\|^2 + \frac{1}{8} \mu \tau \sum_{n=0}^{N-1} \|\nabla e_u^n\|^2. \tag{4.6}
$$

Proof. Multiplying (4.3a) by $2\tau e_\rho^{n+1}$, and integral over Ω, we can see that

$$
\begin{aligned}
& \|e_\rho^{n+1}\|^2 - \|e_\rho^n\|^2 + \|e_\rho^{n+1} - e_\rho^n\|^2 \\
= & -2\tau(\nabla \rho(t_{n+1}) \cdot (\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), e_\rho^{n+1}) - 2\tau(\nabla e_\rho^{n+1} \cdot \mathbf{u}^n, e_\rho^{n+1}) \\
& -2\tau(\nabla \rho(t_{n+1}) \cdot e_u^n, e_\rho^{n+1}) - \tau(\rho(t_{n+1})(\nabla \cdot e_u^n), e_\rho^{n+1}) \\
& -\tau(e_\rho^{n+1}(\nabla \cdot \mathbf{u}^n), e_\rho^{n+1}) + 2\tau(R_\rho^{n+1}, e_\rho^{n+1}).
\end{aligned}
$$

Note that

$$
-2\tau(\nabla e_\rho^{n+1} \cdot \mathbf{u}^n, e_\rho^{n+1}) - \tau(e_\rho^{n+1} \cdot (\nabla \cdot \mathbf{u}^n), e_\rho^{n+1}) = 0.
$$

Applying the Young's inequality and $\|\nabla\cdot\mathbf{v}\|\leq\|\nabla\mathbf{v}\|$, we obtain that

$$\begin{aligned}
&\|e_\rho^{n+1}\|^2-\|e_\rho^n\|^2+\|e_\rho^{n+1}-e_\rho^n\|^2\\
\leq&\tau\|\nabla\rho(t_{n+1})\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\mathbf{u}_t(t)dt\right\|^2+\tau\|e_\rho^{n+1}\|^2\\
&+\tau\|e_u^n\|^2+\tau\|\nabla\rho(t_{n+1})\|_{L^\infty}^2\cdot\|e_\rho^{n+1}\|^2\\
&+\frac{1}{8}\mu\tau\|\nabla e_u^n\|^2+C\tau\|\nabla\rho(t_{n+1})\|_{L^\infty}^2\cdot\|e_\rho^{n+1}\|^2\\
&+\tau\|R_\rho^{n+1}\|^2+\tau\|e_\rho^{n+1}\|^2\\
\leq&C\tau^3+C\tau\|e_\rho^{n+1}\|^2+\chi^{-1}\tau\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau\|\nabla e_u^n\|^2,
\end{aligned}$$

where we have use the assumptions (4.1) and (4.2). Adding up from 0 to $N-1$, we derive

$$\begin{aligned}
&\|e_\rho^N\|^2+\sum_{n=0}^{N-1}\|e_\rho^{n+1}-e_\rho^n\|^2\\
\leq&C\tau^2+C\tau\sum_{n=0}^{N-1}\|e_\rho^{n+1}\|^2+\chi^{-1}\tau\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2.
\end{aligned}$$

According to the discrete Gronwall's inequality, we obtain

$$\|e_\rho^N\|^2+\sum_{n=0}^{N-1}\|e_\rho^{n+1}-e_\rho^n\|^2\leq C\tau^2+\chi^{-1}\tau\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2.$$

Thus, we complete the proof. □

Lemma 4.3. *Assuming that the solution to* (2.1) *satisfies* (4.2) *and the assumption* (4.1) *hold, then for sufficiently small* τ*, there are the following error estimates*

$$\begin{aligned}
&\|\sigma^N e_u^N\|^2+\varepsilon\tau\|\xi^N\|^2+\|\nabla e_\theta^N\|^2+\frac{3}{2}\tau\sum_{n=0}^{N-1}(\mu\|\nabla e_u^{n+1}\|^2+\|\Delta e_\theta^{n+1}\|^2)\\
&+\sum_{n=0}^{N-1}(\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2)\leq C\tau. \qquad (4.7)
\end{aligned}$$

Proof. Taking the inner product of (4.3b) with $2\tau e_u^{n+1}$, (3.1a) with $\tau|e_u^{n+1}|^2$ and (4.3d) with $-2\tau\Delta e_\theta^{n+1}$, we obtain

$$\begin{aligned}
&\|\sigma^{n+1}e_u^{n+1}\|^2-\|\sigma^n e_u^n\|^2+\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+2\mu\tau\|\nabla e_u^{n+1}\|^2\\
&+\|\nabla e_\theta^{n+1}\|^2-\|\nabla e_\theta^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2+2\tau\|\Delta e_\theta^{n+1}\|^2
\end{aligned}$$

$$
\begin{aligned}
=&-2\tau(\rho(t_{n+1})\mathbf{u}(t_{n+1})\cdot\nabla\mathbf{u}(t_{n+1}),e_u^{n+1})+2\tau(\rho^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1},e_u^{n+1})\\
&-\frac{1}{2}\tau(\rho(t_{n+1})(\nabla\cdot\mathbf{u}(t_n))\mathbf{u}(t_{n+1}),e_u^{n+1})+\frac{1}{2}\tau(\rho^{n+1}(\nabla\cdot\mathbf{u}^n)\mathbf{u}^{n+1},e_u^{n+1})\\
&-2\tau(\nabla\xi^{n+1},e_u^{n+1})-2\tau(\Delta\theta(t_{n+1})\nabla\theta(t_{n+1}),e_u^{n+1})\\
&+2\tau(\Delta\theta^{n+1}\nabla\theta^n,e_u^{n+1})+2\tau(R_u^{n+1},e_u^{n+1})-\tau(\nabla\rho^{n+1}\cdot\mathbf{u}^n,|e_u^{n+1}|^2)\\
&-\frac{1}{2}\tau(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),|e_u^{n+1}|^2)+2\tau(\mathbf{u}(t_{n+1})\cdot\nabla\theta(t_{n+1}),\Delta e_\theta^{n+1})\\
&-2\tau(\mathbf{u}^{n+1}\cdot\nabla\theta^n,\Delta e_\theta^{n+1})-2\tau(R_\theta^{n+1},\Delta e_\theta^{n+1}):=\sum_{i=1}^{13}I_i.
\end{aligned}
$$

Rewriting $I_1+I_2+I_3+I_4+I_9+I_{10}$ as follows,

$$
\begin{aligned}
&-2\tau(e_\rho^{n+1}\mathbf{u}(t_{n+1})\cdot\nabla\mathbf{u}(t_{n+1}),e_u^{n+1})-2\tau(\rho^{n+1}\mathbf{u}^n\cdot\nabla e_u^{n+1},e_u^{n+1})\\
&\quad-2\tau(\rho^{n+1}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))\cdot\nabla\mathbf{u}(t_{n+1}),e_u^{n+1})\\
&\quad-2\tau(\rho^{n+1}e_u^n\cdot\nabla\mathbf{u}(t_{n+1}),e_u^{n+1})-\frac{1}{2}\tau(e_\rho^{n+1}(\nabla\cdot\mathbf{u}(t_n))\mathbf{u}(t_{n+1}),e_u^{n+1})\\
&\quad-\frac{1}{2}\tau(\rho^{n+1}(\nabla\mathbf{u}^n)e_u^{n+1},e_u^{n+1})-\frac{1}{2}\tau(\rho^{n+1}(\nabla\cdot e_u^n)\mathbf{u}(t_{n+1}),e_u^{n+1})\\
&\quad-\tau(\nabla\rho^{n+1}\cdot\mathbf{u}^n,|e_u^{n+1}|^2)-\frac{1}{2}\tau(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),|e_u^{n+1}|^2).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-2\tau(\rho^{n+1}\mathbf{u}^n\cdot\nabla e_u^{n+1},e_u^{n+1})-\frac{1}{2}\tau(\rho^{n+1}(\nabla\cdot\mathbf{u}^n)e_u^{n+1},e_u^{n+1})\\
&\quad-\tau(\nabla\rho^{n+1}\cdot\mathbf{u}^n,|e_u^{n+1}|^2)-\frac{1}{2}\tau(\rho^{n+1}(\nabla\cdot\mathbf{u}^n),|e_u^{n+1}|^2)=0.
\end{aligned}
$$

Applying assumptions (4.1), (4.2) and the result of Theorem 4.1, we have

$$
\begin{aligned}
&I_1+I_2+I_3+I_4+I_9+I_{10}\\
\le&\tau\|e_\rho^{n+1}\|^2+\tau\|\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|\nabla\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|e_u^{n+1}\|^2\\
&+\tau\|\rho^{n+1}\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\mathbf{u}_t(t)dt\right\|^2+\tau\|\nabla\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|e_u^{n+1}\|^2\\
&+\tau\|\rho^{n+1}\|_{L^\infty}^2\cdot\|e_u^n\|^2+\tau\|\nabla\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|e_u^{n+1}\|^2\\
&+\tau\|e_\rho^{n+1}\|^2+\frac{1}{8}\tau\|\nabla\cdot\mathbf{u}(t_n)\|_{L^\infty}^2\cdot\|\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|e_u^{n+1}\|^2\\
&+\frac{1}{8}\mu\tau\|\nabla\cdot e_u^n\|^2+C\tau\|\rho^{n+1}\|_{L^\infty}^2\cdot\|\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|e_u^{n+1}\|^2\\
\le&C\tau^3+2\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{4}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&+C\tau\|\sigma^n e_u^n\|^2+C\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{8}\mu\tau\|\nabla e_u^n\|^2.
\end{aligned}
$$

From (4.3c), we deduce that

$$\begin{aligned}I_5=&-2\varepsilon\tau(\zeta^{n+1},\zeta^{n+1}-\zeta^n)+2\varepsilon\tau(\zeta^{n+1},\tilde{P}(t_{n+1})-\tilde{P}(t_n))\\ \leq&-\varepsilon\tau(\|\zeta^{n+1}\|^2-\|\zeta^n\|^2+\|\zeta^{n+1}-\zeta^n\|^2)+\varepsilon^2\tau^2\|\zeta^{n+1}\|^2+C\tau^2.\end{aligned}$$

Rewrite I_6+I_{11} and I_7+I_{12} as follows,

$$\begin{aligned}I_6+I_{11}=&-2\tau(\Delta\theta(t_{n+1})\nabla\theta(t_{n+1}),\mathbf{u}(t_{n+1}))+2\tau(\Delta\theta(t_{n+1})\nabla\theta(t_{n+1}),\mathbf{u}^{n+1})\\ &+2\tau(\mathbf{u}(t_{n+1})\cdot\nabla\theta(t_{n+1}),\Delta\theta(t_{n+1}))-2\tau(\mathbf{u}(t_{n+1})\cdot\nabla\theta(t_{n+1}),\Delta\theta^{n+1})\\ =&2\tau(\Delta\theta(t_{n+1})\cdot\mathbf{u}^{n+1}-\Delta\theta^{n+1}\cdot\mathbf{u}(t_{n+1}),\nabla\theta(t_{n+1})),\\ I_7+I_{12}=&2\tau(\Delta\theta^{n+1}\nabla\theta^n,\mathbf{u}(t_{n+1}))-2\tau(\Delta\theta^{n+1}\nabla\theta^n,\mathbf{u}^{n+1})\\ &-2\tau(\mathbf{u}^{n+1}\cdot\nabla\theta^n,\Delta\theta(t_{n+1}))+2\tau(\mathbf{u}^{n+1}\cdot\nabla\theta^n,\Delta\theta^{n+1})\\ =&2\tau(\Delta\theta^{n+1}\cdot\mathbf{u}(t_{n+1})-\Delta\theta(t_{n+1})\cdot\mathbf{u}^{n+1},\nabla\theta^n),\end{aligned}$$

which implies that

$$\begin{aligned}&I_6+I_7+I_{11}+I_{12}\\ =&2\tau(\Delta\theta(t_{n+1})\cdot\mathbf{u}^{n+1}-\Delta\theta^{n+1}\cdot\mathbf{u}(t_{n+1}),\nabla\theta(t_{n+1})-\nabla\theta^n)\\ =&-2\tau(\Delta\theta(t_{n+1})\cdot e_u^{n+1},\nabla\theta(t_{n+1})-\nabla\theta^n)\\ &\quad+2\tau(\Delta e_\theta^{n+1}\cdot\mathbf{u}(t_{n+1}),\nabla\theta(t_{n+1})-\nabla\theta^n)\\ :=&M_1+M_2.\end{aligned}$$

According to (4.2), we deduce that

$$\begin{aligned}M_1\leq&\tau\|e_u^{n+1}\|^2+\tau\|\Delta\theta(t_{n+1})\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\nabla\theta_t(t)dt\right\|^2\\ &+\tau\|e_u^{n+1}\|^2+\tau\|\Delta\theta(t_{n+1})\|_{L^\infty}^2\cdot\|\nabla e_\theta^n\|^2\\ \leq&C\tau^3+2\tau\|e_u^{n+1}\|^2+C\tau\|\nabla e_\theta^n\|^2,\\ M_2\leq&\frac{1}{8}\tau\|\Delta e_\theta^{n+1}\|^2+C\tau\|\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\nabla\theta_t(t)dt\right\|^2\\ &+\frac{1}{8}\tau\|\Delta e_\theta^{n+1}\|^2+C\tau\|\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|\nabla e_\theta^n\|^2\\ \leq&C\tau^3+\frac{1}{4}\tau\|\Delta e_\theta^{n+1}\|^2+C\tau\|\nabla e_\theta^n\|^2.\end{aligned}$$

Thus

$$I_6+I_7+I_{11}+I_{12}\leq C\tau^3+2\tau\|e_u^{n+1}\|^2+\frac{1}{4}\tau\|\Delta e_\theta^{n+1}\|^2+C\tau\|\nabla e_\theta^n\|^2.$$

Using Lemma 4.1 and Theorem 4.1, we have

$$\begin{aligned}
I_8+I_{13}=&2\tau(R_u^{n+1},e_u^{n+1})-2\tau(R_\theta^{n+1},\Delta e_\theta^{n+1})\\
\le&\frac{1}{C_1}\tau\|R_u^{n+1}\|^2+C\tau\|e_u^{n+1}\|^2+C\tau\|R_\theta^{n+1}\|^2+\frac{1}{4}\tau\|\Delta e_\theta^{n+1}\|^2\\
\le&C\tau^3+\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&\quad+C\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{4}\tau\|\Delta e_\theta^{n+1}\|^2.
\end{aligned}$$

Combining I_1-I_{13}, we get

$$\begin{aligned}
&\|\sigma^{n+1}e_u^{n+1}\|^2-\|\sigma^n e_u^n\|^2+\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+2\mu\tau\|\nabla e_u^{n+1}\|^2\\
&\quad+\varepsilon\tau\|\xi^{n+1}\|^2-\varepsilon\tau\|\xi^n\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2\\
&\quad+\|\nabla e_\theta^{n+1}\|^2-\|\nabla e_\theta^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2+\frac{3}{2}\tau\|\Delta e_\theta^{n+1}\|^2\\
\le&C\tau^3+C\tau^2+3\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{3}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&\quad+C\tau\|\sigma^n e_u^n\|^2+C\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{8}\mu\tau\|\nabla e_u^n\|^2\\
&\quad+\varepsilon^2\tau^2\|\xi^{n+1}\|^2+C\tau\|\nabla e_\theta^n\|^2.
\end{aligned}$$

Adding up from 0 to $N-1$, we obtain

$$\begin{aligned}
&\|\sigma^N e_u^N\|^2+\varepsilon\tau\|\xi^N\|^2+\|\nabla e_\theta^N\|^2+\frac{3}{2}\tau\sum_{n=0}^{N-1}(\mu\|\nabla e_u^{n+1}\|^2+\|\Delta e_\theta^{n+1}\|^2)\\
&\quad+\sum_{n=0}^{N-1}(\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2)\\
\le&C\tau^2+C\tau+C\tau\sum_{n=0}^{N-1}\|\sigma^{n+1}e_u^{n+1}\|^2+\varepsilon^2\tau^2\sum_{n=0}^{N-1}\|\xi^{n+1}\|^2+C\tau\sum_{n=0}^{N-1}\|\nabla e_\theta^n\|^2.
\end{aligned}$$

Applying Gronwall's inequality, we infer that

$$\begin{aligned}
&\|\sigma^N e_u^N\|^2+\varepsilon\tau\|\xi^N\|^2+\|\nabla e_\theta^N\|^2+\frac{3}{2}\tau\sum_{n=0}^{N-1}(\mu\|\nabla e_u^{n+1}\|^2+\|\Delta e_\theta^{n+1}\|^2)\\
&\quad+\sum_{n=0}^{N-1}(\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2)\le C\tau.
\end{aligned}$$

This completes the proof. □

From Lemma 4.3 we can derive that $\|\nabla \mathbf{u}^n\|$, $\|\Delta\theta^n\|\le C$, since $\|\nabla e_u^n\|$, $\|\Delta e_\theta^n\|\le C$, $\forall 0\le n\le N$.

Theorem 4.2. *Assuming that the solutions to* (2.1) *satisfies* (4.2) *and the assumption* (4.1) *hold, then for sufficiently small* τ*, there are the following error estimates*

$$\begin{aligned}&\|\sigma^N e_u^N\|^2+\varepsilon\tau\|\xi^N\|^2+\|\nabla e_\theta^N\|^2+\tau\sum_{n=0}^{N-1}\Big(\mu\|\nabla e_u^{n+1}\|^2+\|\Delta e_\theta^{n+1}\|^2\Big)\\&\quad+\sum_{n=0}^{N-1}(\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2)\\&\le C(\tau^2+\varepsilon^2\tau^2).\end{aligned}$$

Proof. Proceeding the procedure in the proof of Lemma 4.3, we have

$$\begin{aligned}&\|\sigma^{n+1}e_u^{n+1}\|^2-\|\sigma^n e_u^n\|^2+\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+2\mu\tau\|\nabla e_u^{n+1}\|^2\\&\quad+\varepsilon\tau\|\xi^{n+1}\|^2-\varepsilon\tau\|\xi^n\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2\\&\quad+\|\nabla e_\theta^{n+1}\|^2-\|\nabla e_\theta^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2+\frac{3}{2}\tau\|\Delta e_\theta^{n+1}\|^2\\&\le C\tau^3+3\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{3}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\&\quad+C\tau\|\sigma^n e_u^n\|^2+C\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{8}\mu\tau\|\nabla e_u^n\|^2\\&\quad+C\tau\|\nabla e_\theta^n\|^2+2\varepsilon\tau(\xi^{n+1},\tilde{P}(t_{n+1})-\tilde{P}(t_n)).\end{aligned}\tag{4.8}$$

We estimate the last term by using the method inspired in [28]. Consider the decomposition [12]:

$$\mathbf{V}=\mathbf{V}_0\oplus\mathbf{V}_0^\perp,\quad \text{where }\ \mathbf{V}_0^\perp=\{-\Delta^{-1}\nabla q: q\in L^2(\Omega)\},$$

and $v=-\Delta^{-1}\nabla q$ if and only if $-\Delta v=\nabla q$ with $v|_{\partial\Omega}=0$. It is well know that the divergence operator is an isomorphism operator from $\mathbf{V}_0^\perp$ to L_0^2, thus, there exists a unique $\phi(t)\in\mathbf{V}_0^\perp$ such that $\nabla\cdot\phi(t)=\tilde{P}_t$ and $\nabla\cdot\phi_t(t)=\tilde{P}_{tt}$ with

$$\|\nabla\phi(t)\|\le C\|\tilde{P}_t(t)\|,\quad \|\nabla\phi_t(t)\|\le C\|\tilde{P}_{tt}(t)\|,$$

for $\tilde{P}_t(t),\tilde{P}_{tt}(t)\in L_0^2(\Omega)$. Then, we have

$$2\varepsilon\tau\Big(\xi^{n+1},\int_{t_n}^{t_{n+1}}\tilde{P}_t(t)dt\Big)=2\varepsilon\tau\Big(\xi^{n+1},\int_{t_n}^{t_{n+1}}\nabla\cdot\phi(t)dt\Big).$$

Taking the inner product of (4.3b) by $2\varepsilon\tau\int_{t_n}^{t_{n+1}}\phi(t)dt$, we get

$$\begin{aligned}
&2\varepsilon\tau\left(\zeta^{n+1},\int_{t_n}^{t_{n+1}}\nabla\cdot\phi(t)dt\right)\\
=&2\varepsilon\left(\rho^n(e_u^{n+1}-e_u^n),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)+2\varepsilon\tau\left(\rho(t_{n+1})\mathbf{u}(t_{n+1})\cdot\nabla\mathbf{u}(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&-2\varepsilon\tau\left(\rho^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)+\frac{1}{2}\varepsilon\tau\left(\rho(t_{n+1})(\nabla\cdot\mathbf{u}(t_n))\mathbf{u}(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&-\frac{1}{2}\varepsilon\tau\left(\rho^{n+1}(\nabla\cdot\mathbf{u}^n)\mathbf{u}^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)+2\varepsilon\tau\left(\Delta\theta(t_{n+1})\nabla\theta(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&-2\varepsilon\tau\left(\Delta\theta^{n+1}\nabla\theta^n,\int_{t_n}^{t_{n+1}}\phi(t)dt\right)+2\varepsilon\mu\tau\left(\nabla e_u^{n+1},\int_{t_n}^{t_{n+1}}\nabla\phi(t)dt\right)\\
&-2\varepsilon\tau\left(R_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right):=\sum_{i=1}^{9}J_i.
\end{aligned}$$

Rewrite J_1 as

$$\begin{aligned}
J_1=&2\varepsilon\left(\rho^{n+1}e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)-2\varepsilon\left(\rho^n e_u^n,\int_{t_{n-1}}^{t_n}\phi(t)dt\right)\\
&+2\varepsilon\left(\rho^n e_u^n,\int_{t_{n-1}}^{t_n}\phi(t)dt-\int_{t_n}^{t_{n+1}}\phi(t)dt\right)+2\varepsilon\left((\rho^n-\rho^{n+1})e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right).
\end{aligned}$$

From the assumptions on $\tilde{P}_t$ and $\tilde{P}_{tt}$, we have ϕ, $\phi_t\in C(0,T;\mathbf{H}_0^1)$. Thus, there exist $\alpha_n\in[t_n,t_{n+1}]$, $\alpha_{n-1}\in[t_{n-1},t_n]$ and $\widetilde{\alpha}_n\in[t_{n-1},t_{n+1}]$ such that

$$\begin{aligned}
&2\varepsilon\left(\rho^n e_u^n,\int_{t_{n-1}}^{t_n}\phi(t)dt-\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
=&-2\varepsilon\tau(\rho^n e_u^n,\phi(\alpha_n)-\phi(\alpha_{n-1}))=-2\varepsilon\tau(\alpha_n-\alpha_{n-1})(\rho^n e_u^n,\phi_t(\widetilde{\alpha}_n))\\
\le&4\varepsilon\tau^2|(\rho^n e_u^n,\phi_t(\widetilde{\alpha}_n))|\le 2\tau\|\sigma^n e_u^n\|^2+C\varepsilon^2\tau^3.
\end{aligned}$$

For the last term of J_1, using the inequality

$$\|\mathbf{v}\|_{L^3}\le C\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}},$$

we have

$$\begin{aligned}
&2\varepsilon\left((\rho^n-\rho^{n+1})e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
=&2\varepsilon\left((e_\rho^{n+1}-e_\rho^n)e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)-2\varepsilon\left(\left(\int_{t_n}^{t_{n+1}}\rho_t(t)dt\right)e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
\le&C\varepsilon\tau\|e_\rho^{n+1}-e_\rho^n\|\cdot\|\sigma^{n+1}e_u^{n+1}\|^{\frac{1}{2}}\cdot\|\nabla e_u^{n+1}\|^{\frac{1}{2}}\cdot\|\nabla\phi(\alpha_n)\|\\
&+C\varepsilon\tau\left\|\int_{t_n}^{t_{n+1}}\rho_t(t)dt\right\|\cdot\|\nabla e_u^{n+1}\|\cdot\|\nabla\phi(\alpha_n)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \tau\|e_\rho^{n+1}-e_\rho^n\|^2+C\varepsilon^4\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2\\
&\quad +C\varepsilon^2\tau\left\|\int_{t_n}^{t_{n+1}}\rho_t(t)dt\right\|^2+\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2\\
&\leq C\tau^3+C\varepsilon^2\tau^3+\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&\quad +C\varepsilon^4\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{12}\mu\tau\|\nabla e_u^{n+1}\|^2.
\end{aligned}
$$

Rewrite J_2+J_3 as

$$
\begin{aligned}
J_2+J_3=&2\varepsilon\tau\left(e_\rho^{n+1}\mathbf{u}(t_{n+1})\cdot\nabla\mathbf{u}(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&+2\varepsilon\tau\left(\rho^{n+1}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))\cdot\nabla\mathbf{u}(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&+2\varepsilon\tau\left(\rho^{n+1}\mathbf{u}^n\cdot\nabla e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&+2\varepsilon\tau\left(\rho^{n+1}e_u^n\cdot\nabla\mathbf{u}(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right).
\end{aligned}
$$

According to Theorem 4.1 and Lemma 4.3, we deduce that

$$
\begin{aligned}
J_2+J_3\leq&\tau\|e_\rho^{n+1}\|^2+4\varepsilon^2\tau\|\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\|\nabla\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\phi(t)dt\right\|^2\\
&+\varepsilon^2\tau\|\rho^{n+1}\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\mathbf{u}_t(t)dt\right\|^2\cdot\|\nabla\mathbf{u}(t_{n+1})\|_{L^\infty}^2+\tau\left\|\int_{t_n}^{t_{n+1}}\phi(t)dt\right\|^2\\
&+\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2+C\varepsilon^2\tau\|\rho^{n+1}\|_{L^\infty}^2\cdot\|\nabla\mathbf{u}^n\|^2\cdot\left\|\int_{t_n}^{t_{n+1}}\nabla\phi(t)dt\right\|^2\\
&+\tau\|\rho^{n+1}\|_{L^\infty}^2\cdot\|e_u^n\|^2+\varepsilon^2\tau\|\nabla\mathbf{u}(t_{n+1})\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\phi(t)dt\right\|^2\\
\leq&C\tau^3+C\varepsilon^2\tau^3+\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&+C\tau\|\sigma^n e_u^n\|^2+\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2.
\end{aligned}
$$

Similarly, we can infer that

$$
\begin{aligned}
J_4+J_5\leq&C\tau^3+C\varepsilon^2\tau^3+\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&+\frac{1}{24}\mu\tau\|\nabla e_u^n\|^2+\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2.
\end{aligned}
$$

Applying the inequality $\|\nabla^2 v\|\le C\|\Delta v\|$, we get

$$\begin{aligned}
J_6+J_7=&2\varepsilon\tau\left(\Delta e_\theta^{n+1}\nabla\theta(t_{n+1}),\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&+2\varepsilon\tau\left(\Delta\theta^{n+1}\int_{t_n}^{t_{n+1}}\nabla\theta_t(t)dt,\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
&+2\varepsilon\tau\left(\Delta\theta^{n+1}\nabla e_\theta^n,\int_{t_n}^{t_{n+1}}\phi(t)dt\right)\\
\le&\frac{1}{2}\tau\|\Delta e_\theta^{n+1}\|^2+C\varepsilon^2\tau\|\nabla\theta(t_{n+1})\|_{L^\infty}^2\cdot\left\|\int_{t_n}^{t_{n+1}}\phi(t)dt\right\|^2\\
&+\varepsilon^2\tau\|\Delta\theta^{n+1}\|^2\cdot\left\|\int_{t_n}^{t_{n+1}}\Delta\theta_t(t)dt\right\|^2+\tau\left\|\int_{t_n}^{t_{n+1}}\nabla\phi(t)dt\right\|^2\\
&+\frac{1}{4}\tau\|\Delta e_\theta^n\|^2+C\varepsilon^2\tau\|\Delta\theta^{n+1}\|^2\cdot\left\|\int_{t_n}^{t_{n+1}}\nabla\phi(t)dt\right\|^2\\
\le&C\tau^3+C\varepsilon^2\tau^3+\frac{1}{4}\tau\|\Delta e_\theta^n\|^2+\frac{1}{2}\tau\|\Delta e_\theta^{n+1}\|^2.
\end{aligned}$$

Using Lemma 4.1 and Theorem 4.1, we have

$$\begin{aligned}
J_8+J_9\le&\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2+C\varepsilon^2\tau\left\|\int_{t_n}^{t_{n+1}}\nabla\phi(t)dt\right\|^2+\frac{1}{C_1}\tau\|R_u^{n+1}\|^2+C\varepsilon^2\tau\left\|\int_{t_n}^{t_{n+1}}\phi(t)dt\right\|^2\\
\le&C\tau^3+C\varepsilon^2\tau^3+\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{1}{8}\mu\tau^2\sum_{n=0}^{n-1}\|\nabla e_u^n\|^2+\frac{1}{24}\mu\tau\|\nabla e_u^{n+1}\|^2.
\end{aligned}$$

Combining J_1-J_9, we derive

$$\begin{aligned}
&\|\sigma^{n+1}e_u^{n+1}\|^2-\|\sigma^n e_u^n\|^2+\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+2\mu\tau\|\nabla e_u^{n+1}\|^2\\
&+\varepsilon\tau\|\xi^{n+1}\|^2-\varepsilon\tau\|\xi^n\|^2+\varepsilon\tau\|\xi^{n+1}-\xi^n\|^2\\
&+\|\nabla e_\theta^{n+1}\|^2-\|\nabla e_\theta^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2+\tau\|\Delta e_\theta^{n+1}\|^2\\
\le&C\tau^3+C\varepsilon^2\tau^3+7\chi^{-1}\tau^2\sum_{n=0}^{N-1}\|\sigma^n e_u^n\|^2+\frac{7}{8}\mu\tau^2\sum_{n=0}^{N-1}\|\nabla e_u^n\|^2\\
&+C\tau\|\sigma^n e_u^n\|^2+C\tau\|\sigma^{n+1}e_u^{n+1}\|^2+\frac{1}{6}\mu\tau\|\nabla e_u^n\|^2\\
&+\frac{5}{24}\mu\tau\|\nabla e_u^{n+1}\|^2+C\tau\|\nabla e_\theta^n\|^2+\frac{1}{4}\tau\|\Delta e_\theta^n\|^2\\
&+2\varepsilon\left(\rho^{n+1}e_u^{n+1},\int_{t_n}^{t_{n+1}}\phi(t)dt\right)-2\varepsilon\left(\rho^n e_u^n,\int_{t_{n-1}}^{t_n}\phi(t)dt\right).
\end{aligned}$$

Adding up from 0 to $N-1$, we can see that

$$\begin{aligned}&\|\sigma^N e_u^N\|^2+\varepsilon\tau\|\zeta^N\|^2+\|\nabla e_\theta^N\|^2+\tau\sum_{n=0}^{N-1}\left(\frac{3}{4}\mu\|\nabla e_u^{n+1}\|^2+\|\Delta e_\theta^{n+1}\|^2\right)\\&+\sum_{n=0}^{N-1}(\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+\varepsilon\tau\|\zeta^{n+1}-\zeta^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2)\\\le& C\tau^2+C\varepsilon^2\tau^2+C\tau\sum_{n=0}^{N-1}\|\sigma^{n+1}e_u^{n+1}\|^2+C\tau\sum_{n=0}^{N-1}\|\nabla e_\theta^n\|^2+2\varepsilon\left(\rho^N e_u^N,\int_{t_{N-1}}^{t_N}\phi(t)dt\right).\end{aligned}$$

For the last term, we have

$$2\varepsilon\left(\rho^N e_u^N,\int_{t_{N-1}}^{t_N}\phi(t)dt\right)\le\frac{1}{2}\|\rho^N e_u^N\|^2+C\varepsilon^2\tau^2.$$

Applying Gronwall's inequality, we deduce that

$$\begin{aligned}&\|\sigma^N e_u^N\|^2+\varepsilon\tau\|\zeta^N\|^2+\|\nabla e_\theta^N\|^2+\tau\sum_{n=0}^{N-1}\left(\mu\|\nabla e_u^{n+1}\|^2+\|\Delta e_\theta^{n+1}\|^2\right)\\&+\sum_{n=0}^{N-1}(\|\sigma^n(e_u^{n+1}-e_u^n)\|^2+\varepsilon\tau\|\zeta^{n+1}-\zeta^n\|^2+\|\nabla e_\theta^{n+1}-\nabla e_\theta^n\|^2)\\\le& C(\tau^2+\varepsilon^2\tau^2).\end{aligned}$$

This completes the proof. □

Remark 4.2. Substitute the result of Theorem 4.2 into (4.6), we can complete the error estimate of density, i.e., $\|e_\rho^N\|\le C\tau$. Moreover, in the Theorem 4.2, we proved that $\|\sigma^N e_u^N\|\le C\tau$, i.e.,

$$\|\sigma^N\mathbf{u}(t_N)-\sigma^N\mathbf{u}^N\|\le C\tau.$$

Actually,

$$\begin{aligned}\|\sigma(t_N)\mathbf{u}(t_N)-\sigma^N\mathbf{u}^N\|\le& C\|(\sigma(t_N)-\sigma^N)\mathbf{u}(t_N)\|+\|\sigma^N e_u^N\|\\\le& C\|e_\rho^N\|+\|\sigma^N e_u^N\|\le C\tau,\end{aligned}$$

which implies that $\sigma^n\mathbf{u}^n$ are order 1 approximations to $\sigma\mathbf{u}$.

5 Numerical results

The stability and convergence rates derived in Section 4 will be tested by a series of numerical experiment in this section. By using the finite element software Freefem++, some effective numerical results had been obtained. We choose the unit circle as the domain to solve the scheme (3.1a)-(3.1d):

$$\Omega=\{(x,y)\in\mathbf{R}^2 : x^2+y^2\leq 1\}.$$

The initial data are taken as:

$$\rho_0=3+x+y, \qquad \mathbf{u}_0=0,$$
$$P_0=0, \qquad \theta_0=2\pi(\cos(x)-\sin(y)).$$

5.1 Evolutions of energy and orientation field

We plot the energy curves with time steps $\tau=10^{-i}$, $(i=1,2,3,4)$ in Fig. 1 to verify the stability result, which is proved in Section 3. Other parameters are chosen as $T=1$, $\varepsilon=0.1$, $\mu=0.1$ and $h=1/50$, where h represents the mesh size. Two enlarged figures of Fig. 1 are drawn in Fig. 2 to show the difference because the curves in Fig. 1 are almost coincident when $i=2,3,4$. It can be observed that the energy curve changes only slightly when the time step is less than 10^{-2}. An energy law is proved in Theorem 3.1. In fact, it is a modified energy so that it is necessary to compare with original energy. We can see that the curves of modified and original energy are almost same in Fig. 3, where (b) is the enlarged view of (a).

Once the variable θ is calculated, we can restored the orientation field by $\mathbf{d}=(\cos(\theta),\sin(\theta))^T$. The evolution of orientation field is shown in Fig. 4. One can observe that the orientation field is restored perfectly by comparing to the simulation results in [3]. This means that it is very feasible to use this computationally cheaper method to calculate the orientation field.

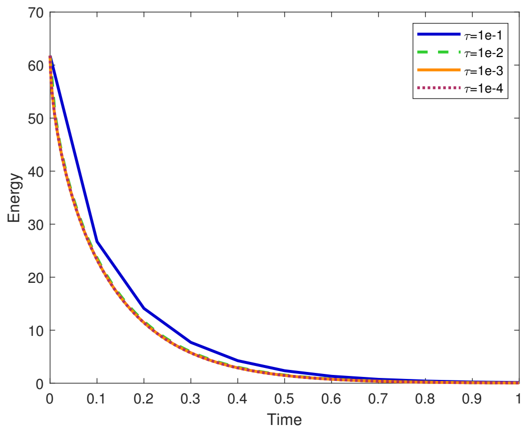
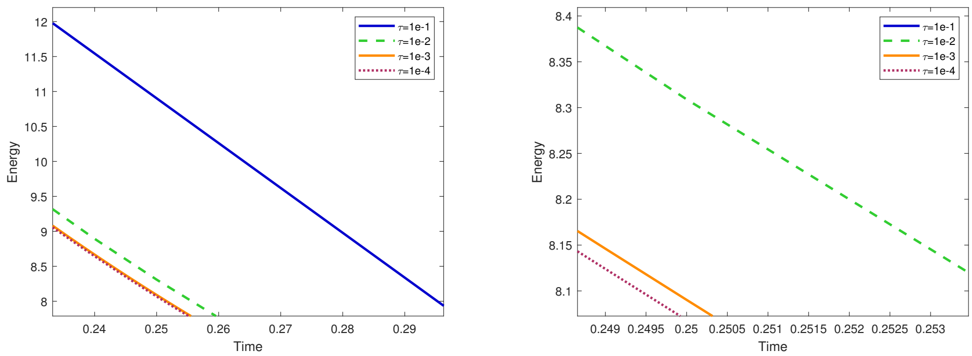
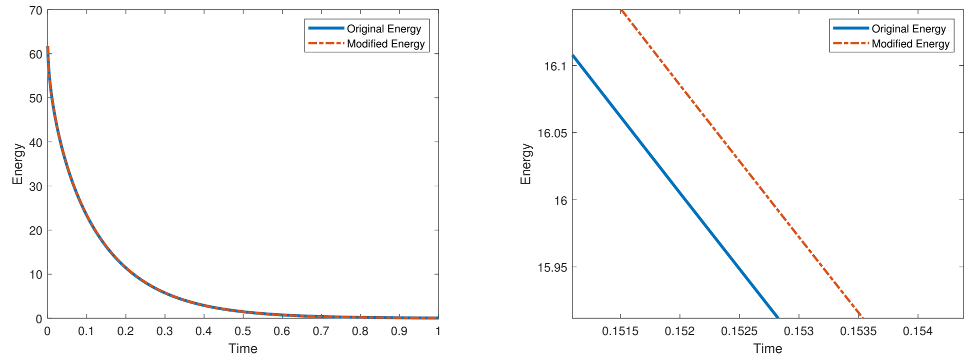


Figure 1: The energy dissipates with different time steps.



(a) (b)

Figure 2: Local enlarged views of Fig. 1.



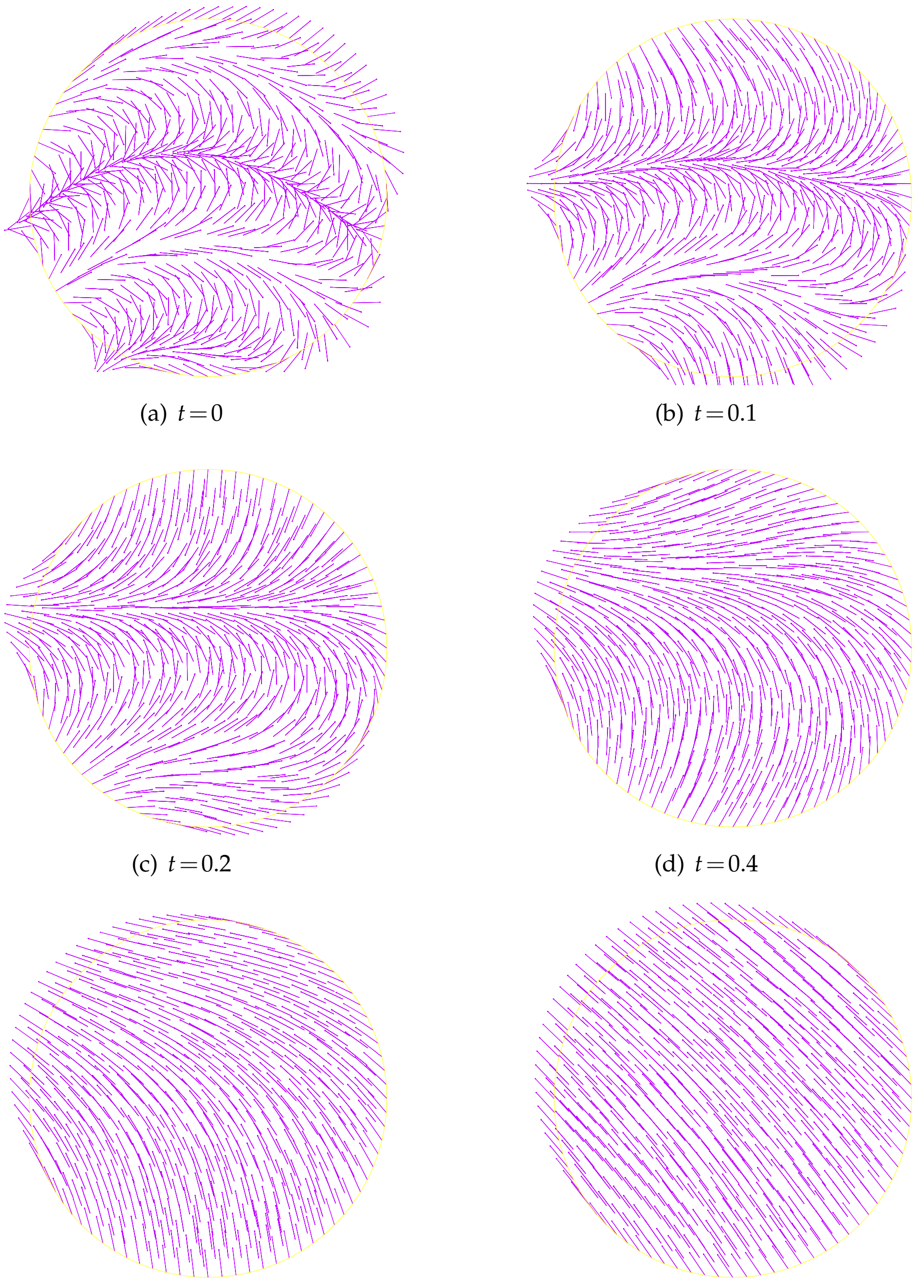
(a) (b)

Figure 3: Comparison between the original and modified energy.

5.2 Convergence rates

Due to the absence of the exact solution, we measure Cauchy error, i.e., the error between two different time step sizes τ and $\tau/2$ is calculated by $\|e_\zeta\| = \|\zeta_\tau - \zeta_{\tau/2}\|$. This is also the reason that there are blanks in the table. We use (P_2, P_2, P_1, P_1) finite element discretization for $(\rho, \mathbf{u}, P, \theta)$. We set the total time $T=1$, the time steps $\tau = 0.05$, 0.025, 0.0125, 0.00625, 0.003125 0.0015625 and 0.00078125, the mesh size $h=1/50$. We choose different viscosities $\mu=1$, 0.1 and penalty parameter $\varepsilon = 0.1$, 0.01 to compare the results.

Tables 1, 2, 3 and 4 show the numerical errors and convergence rates of $(\sigma\mathbf{u},\rho,\theta)$ in L^2, L^2 and H^1, respectively. We can observe that the convergence rate is very much in line with our theoretical analysis in Section 4. We did not obtain the optimal error estimate of pressure since the technical reason. The numerical results of error and convergence of



(a) $t=0$ (b) $t=0.1$

(c) $t=0.2$ (d) $t=0.4$

(e) $t=0.6$ (f) $t=1$

Figure 4: Snapshots of the orientation field.

Table 1: Error and time convergence rates with $\mu=0.1$, $\varepsilon=0.1$.

τ	$\sigma\mathbf{u}-L^2$	rate	$\rho-L^2$	rate	$\theta-H^1$	rate
0.05						
0.025	0.007336		0.013944		0.046176	
0.0125	0.001625	2.174680	0.002853	2.289240	0.022200	1.056610
0.00625	0.000709	1.195560	0.001276	1.161060	0.011003	1.012630
0.003125	0.000334	1.085850	0.000608	1.068420	0.005475	1.007040
0.0015625	0.000162	1.044820	0.000297	1.036240	0.002730	1.003680
0.00078125	0.000080	1.022870	0.000146	1.018610	0.001363	1.001880

Table 2: Error and time convergence rates with $\mu=0.1$, $\varepsilon=0.01$.

τ	$\sigma\mathbf{u}-L^2$	rate	$\rho-L^2$	rate	$\theta-H^1$	rate
0.05						
0.025	0.000751		0.001552		0.045606	
0.0125	0.000165	2.188930	0.000316	2.295930	0.022355	1.028620
0.00625	0.000073	1.168390	0.000144	1.135020	0.011067	1.014340
0.003125	0.000035	1.088340	0.000068	1.072130	0.005504	1.007730
0.0015625	0.000017	1.045860	0.000033	1.037890	0.002744	1.004010
0.00078125	0.000008	1.023370	0.000016	1.019420	0.001370	1.002040

Table 3: Error and time convergence rates with $\mu=0.01$, $\varepsilon=0.1$.

τ	$\sigma\mathbf{u}-L^2$	rate	$\rho-L^2$	rate	$\theta-H^1$	rate
0.05						
0.025	0.007436		0.014090		0.046186	
0.0125	0.001647	2.174290	0.002882	2.289660	0.022197	1.057090
0.00625	0.000720	1.196120	0.001289	1.161250	0.011002	1.012600
0.003125	0.000339	1.085790	0.000614	1.068300	0.005474	1.007030
0.0015625	0.000164	1.044800	0.000300	1.036180	0.002730	1.003670
0.00078125	0.000081	1.022860	0.000148	1.018590	0.001363	1.001880

Table 4: Error and time convergence rates with $\mu=0.01$, $\varepsilon=0.01$.

τ	$\sigma\mathbf{u}-L^2$	rate	$\rho-L^2$	rate	$\theta-H^1$	rate
0.05						
0.025	0.000746		0.001552		0.045606	
0.0125	0.000164	2.188690	0.000316	2.295950	0.022355	1.028620
0.00625	0.000073	1.168440	0.000144	1.135000	0.011067	1.014340
0.003125	0.000034	1.088370	0.000069	1.072130	0.005504	1.007730
0.0015625	0.000016	1.045880	0.000033	1.037890	0.002744	1.004010
0.00078125	0.000008	1.023380	0.000017	1.019420	0.001370	1.002040

Table 5: Error and time convergence rates of pressure.

τ	$\mu=0.1, \varepsilon=0.1$	rate	$\mu=0.1, \varepsilon=0.01$	rate
0.05				
0.025	0.051657		0.052358	
0.0125	0.011453	2.173250	0.011491	2.187920
0.00625	0.005002	1.195180	0.005111	1.168730
0.003125	0.002356	1.086400	0.002404	1.088580
0.0015625	0.001142	1.045070	0.001164	1.045970
0.00078125	0.000562	1.022990	0.000573	1.023420

Table 6: Error and time convergence rates of pressure.

τ	$\mu=0.01, \varepsilon=0.1$	rate	$\mu=0.01, \varepsilon=0.01$	rate
0.05				
0.025	0.052328		0.052427	
0.0125	0.011603	2.173100	0.011506	2.187920
0.00625	0.005066	1.195550	0.005118	1.168720
0.003125	0.002386	1.086350	0.002407	1.088580
0.0015625	0.001156	1.045050	0.001166	1.045970
0.00078125	0.000569	1.022980	0.000573	1.023420

pressure are shown in the Tables 5 and 6. We can see that the convergence rate of pressure in L^2 is nearly first order.

6 Conclusions

In this paper, we construct a numerical scheme for the reformulated Ericksen-Leslie system with variable density. The computational efficiency and stability are improved by using polar coordinate and pressure penalty methods. The main work is prove the first-order temporal convergence rate. In addition, some numerical experiments have verified the theoretical derivation results. We made some assumptions for density instead of proving them, because this is not the focus of this paper. Readers interested in a detailed discussion can refer to [8] and [19]. The spatial and higher order error estimations need to be further extended as in [20].

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