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A Numerical Algorithm for Solving an Inverse Nonlinear Parabolic Problem

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Abstract. In this paper, we propose an algorithm for numerical solving an inverse nonlinear diffusion problem. The algorithm is based on the Laplace transform technique and the finite difference method in conjunction with the least-squares scheme. To regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization method to obtain the stable numerical approximation to the solution. To show the efficiency and accuracy of the present method a test problem will be studied.

Keywords: Inverse nonlinear parabolic problem, Laplace transform, Finite difference method, Least-squares method, Regularization method.

1. Introduction

Inverse heat conduction problems (IHCPs) appear in many important scientific and technological fields. Hence analysis, design implementation and testing of inverse algorithms are also are great scientific and technological interest. Mathematically, the inverse problems belong to the class of problems called the ill-posed problems. That is, their solution does not satisfy the general requirement of existence, uniqueness, and stability under small changes to the input data. To overcome such difficulties, a variety of techniques for solving IHCPs have been proposed.

Numerical solution of an inverse nonlinear diffusion problem requires to determine an unknown diffusion coefficient from an additional information. These new data are usually given by adding small random errors to the exact values from the solution to the direct problem. This paper presents the inverse determination of the diffusion coefficient of an unknown porous medium[1].

Mathematically, IHCPs belong the class of ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. The physical reason for the ill-posedness of the estimation problem is that variations in the surface conditions of the solid body are damped towards the interior because of the diffusive nature of heat conduction. As a consequence, large-amplitude changes at the surface have to be inferred from small-amplitude changes in the measurements data. Errors and noise in the data can therefore be mistaken as significant variations of the surface state by the estimation procedure. Therefore the IHCP has a unique solution, but this solution is unstable. In this paper this instability is overcome using the Tikhonov regularization method with L-curve criterion for the choice of the regularization parameter.

The outline of this paper is as follows. In the section 2, we formulate an inverse nonlinear parabolic problem. In the section 3, we linearize nonlinear term by Taylor's series expansion, remove time-dependent terms by Laplace transform technique, discretize governing equations by finite difference method and used least squares method for correction unknown coefficients. Numerical experiments in section 4, confirm our theoretical results for an unknown porous medium.

2. Mathematical model

The mathematical model of an inverse nonlinear parabolic problem with initial and boundary conditions is the following form

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$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right), 0<x<1, 0<t<T, \tag{1}$$

$$u(x,0)=p(x), 0<x<1, \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x}=g(t), \ 0<t<T, \tag{3}$$

$$\frac{\partial u(1,t)}{\partial x}=q(t), \ 0<t<T, \tag{4}$$

$$u(1,t)=f(t), \ 0<t<T, \tag{5}$$

where T is a given positive constant, and $g(t)$, $p(x)$ and $q(t)$ are piecewise-continuous known functions, while $u(x,t)$ and diffusion coefficient $a(u(x,t))>0$, [2], are unknown which remain to be determined.

For an unknown function $a(u)$ we must therefore provide additional information (5) to provide a unique solution $(u,a(u))$ to the inverse problem (1)-(5). Parabolic problems including equation (1) have been previously treated by many authors who considered certain special case of this type of problem [6-11]. In this article, under certain conditions on $g(t)$, $p(x)$, $q(t)$ and $f(t)$, we shall identify both $u(x,t)$ and diffusion coefficient $\mathrm{a}(\mathrm{u})$ at any time by using the over specified condition (5), initial and boundary conditions (2)-(4).

3. Description of the numerical scheme

Consider the one-dimensional nonlinear problem described by the problem (1)-(5), where (1) is nonlinear. The application of the present numerical method to find the solution of problem (1)-(5), can be divided into the following steps.

3.1. Linearizing the nonlinear terms

Since the application of the Laplace transform technique is only restricted to the linear system, so that the nonlinear term in equation (1) must be linearized. Therefore, we used Taylor's series expansion for linearized nonlinear terms and we obtain [9]

$$\frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial u}\left(k(u)\right)_{u=\bar{u}}\frac{\partial^2 u}{\partial x^2}=a(\bar{u})\frac{\partial^2 u}{\partial x^2}, \tag{6}$$

where

$$k(u)=\int_0^u a(\rho)d\rho, \tag{7}$$

is a nonlinear function and $\bar{u}=(\bar{u}_0,\bar{u}_1,\cdots,\bar{u}_N)$ denotes the previously iterated solution.

3.2. Remove time dependent terms

For remove time dependent terms from equations (2) , (3) , (4) and (6) the method of the Laplace transform is employed. Therefore we obtain

$$a(\bar{u})\frac{\partial^2\tilde{u}}{\partial x^2}=s\tilde{u}-p(x), 0<x<1, \tag{8}$$

$$\frac{\partial\tilde{u}}{\partial x}=G(s), \ x=0, \tag{9}$$

$$\frac{\partial\tilde{u}}{\partial x}=Q(s), \ x=1, \tag{10}$$

where $s = \nu + i\omega;\ \nu,\ \omega \in R$, $\tilde{u},\ \frac{\partial \tilde{u}}{\partial x},\ \frac{\partial^2 \tilde{u}}{\partial x^2},\ Q(s)$ and $G(s)$ are Laplace transform of $u,\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2},\ q(t)$ and $g(t)$ respectively.

3.3. Finite difference method for discretizing

In this step, we use central finite difference approximation for discretizing problem (8) – (10). Therefore

$$a(\bar{u}_\mu)\frac{\tilde{u}_{\mu+1}-2\tilde{u}_\mu+\tilde{u}_{\mu-1}}{h^2}-s\tilde{u}_\mu=-p(\mu h),\ \mu=0,1,\cdots,N, \tag{11}$$

$$\frac{\tilde{u}_1-\tilde{u}_{-1}}{2h}=G(s),\ x=0, \tag{12}$$

$$\frac{\tilde{u}_{N+1}-\tilde{u}_{N-1}}{2h}=Q(s),\ x=1. \tag{13}$$

Problem (11)-(13) may be written in the following matrix form

$$A\tilde{U}=B, \tag{14}$$

where

$$A=\begin{pmatrix} -2a(\bar{u}_0)-sh^2 & 2a(\bar{u}_0) & 0 & 0 \\ a(\bar{u}_1) & -2a(\bar{u}_1)-sh^2 & a(\bar{u}_1) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a(\bar{u}_{N-1}) & -2a(\bar{u}_{N-1})-sh^2 & a(\bar{u}_{N-1}) \\ 0 & 0 & 2a(\bar{u}_N) & -2a(\bar{u}_N)-sh^2 \end{pmatrix},$$

and

$$\tilde{U}^t=(\tilde{u}_0\ \ \tilde{u}_1\ \ \cdots\ \ \tilde{u}_{N-1}\ \ \tilde{u}_N),$$
$$B^t=(b_0\ \ b_1\ \ \cdots\ \ b_{N-1}\ \ b_N),$$

where $b_0=h^2a(\bar{u}_0)p(0)-2hG(s),\ b_i=-h^2p(ih),\ i=1,\cdots,N-1,$ and $b_N=-h^2p(Nh)-2ha(\bar{u}_N)Q(s)$. Note that equation (17) is a linear equation.

Theorem. If A be a $N\times N$ matrix, $N\geq 3$, and $h\geq\frac{1}{\sqrt{|s|}}$, where $s=\nu+i\omega$ is Laplace parameter, then the finite difference scheme (14) is stable.

Proof. From equation (14) obtains

$$\tilde{U}=A^{-1}B.$$

The matrix determining the propagation of the error in the above system is A^{-1}. Therefore difference scheme (14) will be stable when the modulus of every eigenvalue of A^{-1} does not exceed one. The matrix A can be written as

$$A=\sigma\begin{pmatrix} -2 & 2 & & & & \\ 1 & -2 & 1 & & & \\ & & & & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{pmatrix}+\begin{pmatrix} -sh^2 & & & & & \\ & -sh^2 & & & & \\ & & -sh^2 & & & \\ & & \cdot & \cdot & \cdot & \\ & & & & -sh^2 & \\ & & & & & -sh^2 \end{pmatrix},$$

where σ is the maximum value of $a(\bar{u}_i)$, $i = 0(1)N-1$,. Therefore eigenvalues of A are

$$\lambda_m = -4\sigma \sin^2\left(\frac{(m-1)\pi}{2(N-1)}\right) - sh^2,$$

where $m = 1,...,N$, Since $a(\bar{u}_i) > 0$, then clearly the modulus of every λ_m, greater than one for all $h \geq \frac{1}{\sqrt{|s|}}$. Hence the modulus of every eigenvalue of A^{-1} less than one with this condition over h.

The Gaussian elimination algorithm is used to solve $\tilde{U}$ and the numerical inversion of the Laplace transform technique ([12] – [13]) is applied to invert the transformed result to the physical quantity $U^t = (u_0 \quad u_1 \quad ... \quad u_{N-1} \quad u_N)$. These updated values of U are used to calculate A and B for iteration. This computational procedure is performed repeatedly until desired convergence is achieved. The unknown function $a(u)$ is difficult to be approximated by a polynomial function for the whole time domain considered. Therefore the time domain $t_0 \leq t \leq T$ will be divided into some intervals where t0 is the initial measurement time. Each of the intervals is assumed to be $t_{m-1} \leq t \leq t_m$ where $t_m = t_0 + m\Delta t$, $m = 1,...,N$ and $\Delta t = \frac{T - t_0}{N}$. In this work the polynomial form proposed for the unknown $a(u)$ before performing the inverse calculation. Therefore $a(u)$ approximated as

$$a(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_q u^q, \tag{15}$$

where $\{a_0, a_1, a_2, ..., a_q\}$, are constants which remain to be determined simultaneously.

3.4. Least-squares minimization technique

To minimize the sum of the squares of the deviations between $u_N(t)$ (calculated) and $f(t)$ at the specific times $t = t_r$, we use least squares method. The error in the estimate is

$$E(a_0, a_1, a_2, ..., a_q) = \sum_{j=1}^{N} \left(u_N(t_j) - f(t_j)\right)^2, \tag{16}$$

which remain to be minimized. The estimated values of a_i, are determined until the value of $E(a_0, a_1, a_2, ..., a_q)$, is minimum. The computational procedure for estimating unknown coefficients a_i is well addressed in [9], therefore the correction linear system corresponding to the values of a_i can be expressed as

$$\Lambda\Theta = C, \tag{17}$$

where

$$\Lambda = \begin{pmatrix} \sum_{j=1}^{N}(Y_j^0)^2 & \sum_{j=1}^{N} Y_j^0 Y_j^1 & \cdots & \sum_{j=1}^{N} Y_j^0 Y_j^q \\ \sum_{j=1}^{N} Y_j^0 Y_j^1 & \sum_{j=1}^{N}(Y_j^1)^2 & \cdots & \sum_{j=1}^{N} Y_j^1 Y_j^q \\ & & & \\ \sum_{j=1}^{N} Y_j^0 Y_j^q & \sum_{j=1}^{N} Y_j^1 Y_j^q & \cdots & \sum_{j=1}^{N}(Y_j^q)^2 \end{pmatrix}$$

$$C = \left(-\sum_{j=1}^{N} Y_j^0 e_j \quad \cdots \quad -\sum_{j=1}^{N} Y_j^q e_j\right)^T,$$

$$\Theta = \begin{pmatrix} h_0 & h_1 & \cdots & h_q \end{pmatrix}^T, \quad e_j = u_N(t_j) - f(t_j),$$

$$Y_j^i = \frac{\partial u_N(t_j)}{\partial a_j}, \quad i = 0, \dots, q, \quad j = 1, \dots, N,$$

and h_i denotes the correction for initial values of a_i. The Tikhonov regularized solution ([3]−[4]−[5]) to the system of linear algebraic equation

$$\Lambda\Theta = C,$$

is given by

$$\Theta_\alpha : \phi_\alpha(\Theta_\alpha) = \min_\Theta \phi_\alpha(\Theta_\alpha),$$

where ϕ_α represents the zeroth order Tikhonov functional given by

$$\phi_\alpha(\Theta) = \|\Lambda\Theta - C\|^2 + \alpha^2\|\Theta\|^2,$$

Solving $\nabla\phi_\alpha(\Theta) = 0$ with respect to Θ, then we obtain, the Tikhonov regularized solution of the regularized equation

$$\varphi_\alpha = \left(\Lambda^T\Lambda + \alpha^2 I\right)^{-1}\Lambda^T C.$$

Definition. Let $A \in R^{m\times n}$ be a matrix and let $m \geq n$, then the singular value decomposition (SVD) of A is defined by

$$A = U\begin{pmatrix} S \\ 0 \end{pmatrix}V^T,$$

where $U = \begin{pmatrix} u_1 & \dots & u_m \end{pmatrix} \in R^{m\times m}$, and $V = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \in R^{n\times n}$, are orthogonal matrix with orthogonal columns, and u_i, v_i are the left and right singular vectors of A respectively, and $S \in R^{n\times n}$ is a diagonal matrix that diagonal elements appearing in non-increasing order of nonnegative singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. In case $m < n$ then define the SVD from A^T.

Now for $\Lambda\theta = C$,

$$\Lambda = \sum_{i=1}^{\gamma+1} u_i \sigma_i v_i^T,$$

$$\theta = \sum_{i=1}^{\gamma+1} \left(v_i^T\theta\right) v_i,$$

and

$$C = \sum_{i=1}^{\gamma+1} \left(u_i^T B\right) u_i.$$

Therefore the Tikhonov solution can be formulated as

$$\theta_\alpha = \left(\Lambda^T\Lambda + \alpha^2 I\right)^{-1}\Lambda^T B = \sum_{i=1}^{\gamma+1} \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2}\frac{u_i^T C}{\sigma_i} v_i.$$

In our computation we use the L-curve scheme to determine a suitable value of α ([3]−[5]).

Note that, the L-curve method is sketched in the following form,

$$L = \left\{\left(\log\left(\|\theta_\alpha\|^2\right)\right), \left(\log\left(\|\Lambda\theta_\alpha - C\|^2\right)\right), \ \alpha > 0\right\}, \tag{15}$$

The curve is known as L-curve and a suitable regularization parameter α corresponds to a regularized solution near the, corner, of the L-curve [3].

4. Numerical experiment

In this section the stability and accuracy of the scheme presented in section 3 is evaluated. All the computations are performed on the PC (pentium(R) 4 CPU 3.20 GHz).

Example. In this example, let us consider the following inverse nonlinear parabolic problem

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right),\ 0<x<1,\ 0<t<T, \tag{18}$$

$$u(x,0)=x,\ 0<x<1, \tag{19}$$

$$\frac{\partial u(0,t)}{\partial x}=1,\ 0<t<T, \tag{20}$$

$$\frac{\partial u(1,t)}{\partial x}=1,\ 0<t<T, \tag{21}$$

$$u(1,t)=t+1,\ 0<t<T, \tag{22}$$

with unique exact solution

$$a(u)=1+u,\ u(x,t)=x+t.$$

Therefore to solve the problem (18)–(22) the unknown coefficient $a(u)$ defined as the following form

$$a(u)=a_0+a_1u.$$

For determine a_0 and a_1 we use

$$E(a_0,a_1)=\sum_{j=1}^{N}\left(u_N(t_j)-f(t_j)\right)^2,$$

therefore the coefficients can be obtained.

Tables 1 and 2, respectively, are shown the values of $u_{i,j}$ in $x=ih$ and $t=jk$ when $k=\frac{1}{10}$, $h=\frac{1}{6}$, $\tau_i=0.04$. The estimated values of a_0, a_1 are $a_0=1.001821$ and $a_1=1.025121$.

Table 1.*The value of* $u_{i,j}$ *when* $i=0,1,2$ *and* $j=1,\dots,5$

	Numerical	*Exact*	*Numerical*	*Exact*	*Numerical*	*Exact*
j	$u_{0,j}$	$u_{0,j}$	$u_{1,j}$	$u_{1,j}$	$u_{2,j}$	$u_{2,j}$
1	0.104774	0.1	0.263512	0.266667	0.433299	0.433333
2	0.197993	0.2	0.359834	0.366667	0.533021	0.533333
3	0.299851	0.3	0.466451	0.466667	0.631030	0.633333
4	0.397748	0.4	0.559884	0.566667	0.734412	0.733333
5	0.505412	0.5	0.665218	0.666667	0.832951	0.833333

Table 2.*The value of* $u_{i,j}$ *when* $i = 3,4,5$ *and* $j = 1,...,5$

	Numerical	*Exact*	*Numerical*	*Exact*	*Numerical*	*Exact*
j	$u_{3,j}$	$u_{3,j}$	$u_{4,j}$	$u_{4,j}$	$u_{5,j}$	$u_{5,j}$
1	0.599465	0.6	0.762469	0.766667	0.933941	0.933333
2	0.698851	0.7	0.863478	0.866667	1.031386	1.033333
3	0.802003	0.8	0.958819	0.966667	1.134651	1.133333
4	0.900641	0.9	1.065950	1.066667	1.253264	1.233333
5	1.001521	1.0	1.166407	1.166667	1.324871	1.333333

Figures 1 and 2 show the comparison between the exact results and the present numerical results for $u_{i,j}$ in $x = ih$ and $t = jk$ when $k = \frac{1}{10}$, $h = \frac{1}{6}$, $\tau_i = 0.04$.

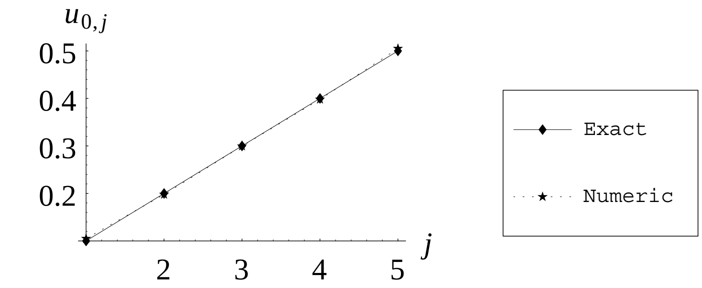

Figure 1. Comparison between the exact results and the present numerical results of the problem (18)-(22).

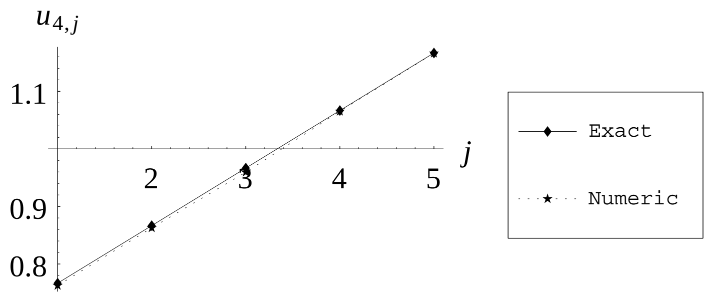

Figure 2. Comparison between the exact results and the present numerical results of the problem (18)-(22).

5. Conclusion

A numerical method to estimate unknown coefficient is proposed for an inverse nonlinear parabolic problem and the following results are obtained.

1. The present study, successfully applies the numerical method involving the Laplace transform technique and the finite difference method in conjunction with the least-squares scheme to an IHCP.

2. From the illustrated examples it can be seen that the proposed numerical method is efficient and accurate to estimate the thermal diffusivity in a one-dimensional nonlinear inverse diffusion problem..

3. Owing to the application of the Laplace transform, the present method is not a time-stepping procedure. Thus the unknown diffusion coefficient at any specific time can be predicted without any step-by-step computations from $t = t_0$. We also apply other different sets of the initial guesses, such as $\{a_0, a_1, a_2, ..., a_q\} = \{0.3, 0.3, ..., 0.3\}$ and $\{1.2, 1.2, ..., 1.2\}$, results show that the effect of the initial guesses on the accuracy of the estimates is not significant for the present method.

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