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A Full-Newton Step Primal-Dual Interior Point Algorithm for Linear Complementarity Problems*

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Abstract. In this paper, we give a full-Newton step primal-dual interior-point algorithm for monotone horizontal linear complementarity problem. The searching direction is obtained by modification of the classic Newton direction, and which also enjoys the quadratically convergent property in the small neighborhood of central path. The complexity bound is derived, which is $\mathrm{O}\left(2\sqrt{n}\log\frac{n\mu^0}{\varepsilon}\right)$.

Keywords: horizontal linear complementarity problem, interior-point algorithm, full-Newton step, complexity bound.

1. Introduction

A monotone horizontal linear complementarity problem (LCP) is to find a pair $x, s \in R^n$ such that

$$Mx + Ns = q, \quad xs = 0, \quad x, s \geq 0. \tag{1}$$

where $q \in R^m$ and $M, N \in R^{m\times n}$, moreover M and N have the column monotonic property, i.e., for any $u, w \in R^n$

$$Mu + Nw = 0 \Rightarrow u^T w \geq 0. \tag{2}$$

The formulation (1) includes linear and convex quadratic programming problems expressed by their optimality conditions in their usual format. Properties of this formulation are described in [1], where the rank $r([M, N]) = n$ has been proved under the monotonic hypothesis.

There are a variety of solution approaches for LCP which have been studied intensively. Among them, the interior-point methods (IPMs) gained much attention than other methods. Due to the close connection between LCP and linear and convex quadratic programming problems, some IPMs for linear and convex quadratic programming problems have been extended to LCP. For instance, Gonzaga et al. [2,3] studied the largest step path following algorithm for LCP and showed that the fast convergence of the simplified largest step path following algorithm. Huang et al. [4] proposed a high-order feasible IPM for LCP with $\mathrm{O}\left(\sqrt{n}\log\frac{\varepsilon^0}{\varepsilon}\right)$ iterations. Monteiro et al. [5] studied the limiting behavior of the derivatives of certain trajectories associated with the monotone LCP. Zhang [6] presented a class of infeasible IPMs for LCP and showed that the algorithm has $\mathrm{O}\left(n^2\log\frac{1}{\varepsilon}\right)$ under some mild assumptions. Some other relevant references can be found in [4,7,8].

In this paper, we give a full-Newton step IPM for LCP, the algorithm uses a modified Newton direction, which enjoys the nice property of quadratically convergent in the small neighborhood of central path. We

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derive the complexity bound for the algorithm, and the complexity result is the best-known for LCP.

The paper is organized as follows: In Section 2, the basic concepts of IPMs are given, which include central path and the classic Newton direction. In section 3, we give a scaled version of the classic Newton direction, and from which we give a modified Newton direction. The generic algorithm is described in section 4. In section 5, the properties of full-Newton step are analyzed, which include the estimation of the upper bound for dual gap and the increase of the proximity after one full-Newton step, the decrease of proximity after the parameter update is also given in this section. At the end of this section, we give a complexity result for the full-Newton step IPM. Section 6 gives a simple numerical example. Section 7 ends the paper with a conclusion.

Some notations used throughout the paper are as follows. $\|\cdot\|$ and $\|\cdot\|_\infty$ denotes the 2-norm and ∞-norm of a vector respectively. For any $x=(x_1,x_2,\ldots,x_n)^T\in R^n$, $x_{\min}$ denotes the smallest value of the components of x and xs denotes the componentwise (or Hadamard) product of the vectors x and s.

2. Preliminary

We assume the following hypotheses hold: the existence of an interior feasible solution and the existence of a strictly complementarity optimal solution.

2.1. The central path

The basic idea of the IPM is to replace the second equation in (1) by the parameterized equation $xs=\mu e$, with $\mu>0$. Thus we consider the system

$$Mx+Ns=q,\quad xs=\mu e,\quad x,s\geq 0. \tag{3}$$

The parameterized system (3) has a unique solution for each $\mu>0$. This solution is denoted as $(x(\mu),s(\mu))$ and is called the μ-center of LCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of LCP. If $\mu\to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $xs=0$, the limit yields an optimal solution for LCP, see [6,9].

2.2. The classic Newton direction

The search directions used in all primal-dual IPMs were computed from the linear system

$$\begin{aligned} M(x+\Delta x)+N(s+\Delta s) &= q \\ (x+\Delta x)(s+\Delta s) &= \mu e \end{aligned} \tag{4}$$

Neglecting the quadratic term $\Delta x\Delta s$ in the left-hand side expression of the second equation, we obtain the so-called classic Newton direction Δx and Δs.

$$\begin{aligned} M\Delta x+N\Delta s &= 0 \\ x\Delta s+s\Delta x &= \mu e-xs \end{aligned} \tag{5}$$

The unique solution of the system (5) is guaranteed by Lemma 4.1 in [9].

3. New search direction

To describe the ideas underlying this paper, we need to consider a scaled version of the system (5) that defines the search directions.

3.1. A scaled-Newton direction

Now we introduce the scaled vector v and the scaled search directions d_x and d_s according to

$$v=\sqrt{\frac{xs}{\mu}} \text{ and } d_x=\frac{v\Delta x}{x},\ d_s=\frac{v\Delta s}{s} \tag{6}$$

According to (6), the system (5) can be rewritten as