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Linear Variational Inequality Model of L1-norm Minimization Problems with Applications to Compressive Sensing

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Abstract. This paper considers the l_1 norm minimization (L1NM) problem which is a well known problem in compressive sensing. We first transform the L1NM problem into a bounded-constrained quadratic programming (BCQP), and then into an equivalent linear variational inequality (LVI) problem. To solve the resulting LVI problem, a modified extra-gradient method proposed by Han is introduced, whose global convergence can be guaranteed by Han's paper. The method is easily performed, since it only make a projection to the nonnegative orthant and calculate some matrix-vector products to get the next iterate. Numerical simulations are conducted to verify the efficiency of the proposed method.

Keywords: l_1 norm minimization problem; linear variational inequality problem; compressive sensing.

1. Introduction

In this paper we consider the following l_1 norm minimization problem, denoted by L1NM problem,

$$\min_{x\in R^n} \frac{1}{2}\|Ax-b\|_2^2 + \mu\|x\|_1 \tag{1}$$

where $A \in R^{m\times n}(m << n)$ is a linear operator, $b \in R^m$ is an observation, $x \in R^n$ is the vector of unknowns, $\|x\|_1 = \sum_{i=1}^{n}|x_i|$ is the l_1 norm of x, and parameter $\mu > 0$ is used to trade off both terms for minimization, whose value is quite important, for example, if μ is too large then the solution is the trivial one: $x = 0$. Model (1) mainly appeared in statistical and signal processing fields, in which a sparse original signal $\bar{x} \in R^n$ is desirable to be recovered by solving the L1NM problem. In fact, preliminary work in this area showed that if the original signal is sparse or approximately sparse in some orthogonal basis, an exact restoration can be produced via solving (1) so long as certain conditions, such as the Restricted Isometry Property (RIP) hold [1,2].

In recent years, the study in numerical methods for (1) has taken good progress, and many efficient iterative algorithms have been proposed, analyzed, and tested. Among them, the most popular methods are the iterative shrinkage/thresholding (IST) type methods, including IST fixed-point continuation algorithm (FPC) in [3], two-step IST (TwIST) in [4], the fast IST algorithm (FISTA) in [5], and the latter two algorithms have virtually the same complexity as IST, but have better convergence performance. Gradient based algorithm is also quite efficient for solving the L1NM problem due to its simplicity. Gradient projection method for sparse reconstruction (GPSR) proposed by Figueiredo et al.[6] first transform the L1NM problem to a bound-constrained quadratic programming (BCQP) by splitting x and solves BCQP using Barzilai-Borwein gradient method with an efficient nonmonotone line search. Other gradient based methods can be found in [7,8]. In addition, the alternating direction method (ADM) algorithms are also introduced to solve the L1MN problem. For example, Yang and Zhang [9] investigates the L1MN problem from either the primal or the dual forms and solves some l_1 regularized problems related to L1MN problem.

In this paper, we continue to study the L1MN problem based on its BCQP transformation. As is pointed by Xiao et al.[10], the resulting BCQP is equivalent to a linear variational inequality (LVI) problem. To our knowledge, researchers haven't investigate the iterative method for the L1MN problem based on its LVI formulation. Here, the resulting LVI problem is solved by the modified extra-gradient method proposed by Han [11], which is an efficient method with quite low computational load. Therefore, the global convergence is followed directly in this literature. To do so, the rest of the paper is organized as follows. Section 2, we summarize some basic definitions used in the paper, and list the steps of our algorithm. In Section 3, we present and analyze the experimental results, which indicate that the proposed algorithm is quite efficient. Finally, we summarize our paper in Section 4.

2. Preliminaries and the algorithm

In this section, we briefly review some related knowledge, and state our algorithm.

Firstly, we give the definition of projection operator, which is defined as a mapping from R^n to its nonempty closed convex subset Ω:

$$P_{\Omega}[x] := \arg\min\{\|y-x\| \mid y \in \Omega\,|, \forall x \in R^n.$$

In [6], Figueiredo et al. express the L1NM problem as a quadratic programming by splitting the variable x into its positive and negative parts. That is, for any vector $x \in R^n$, it can be formulated for

$$x = u - v, u \geq 0, v \geq 0,$$

where $u \in R^n, v \in R^n$, and $u_i = (x_i)_+, v_i = (-x_i)_+$ for all $i = 1,2,\ldots,n$ with $(\cdot)_+ = \max\{0,|\cdot|\}$. We thus have $\|x\|_1 = e_n^T u + e_n^T v$, where e_n is an n-dimensional vector with all elements one, so the L1NM problem (1) can be written as the following bound-constrained quadratic programming (BCQP):

$$\min_{u,v} \frac{1}{2}\|y - A(u-v)\|_2^2 + \mu e_n^T u + \mu e_n^T v$$
$$s.t. u \geq 0, v \geq 0.$$

Then, the above problem is further written in more standard BCQP form:

$$\min_{u,v} \frac{1}{2} z^T H z + c^T z$$
$$s.t. z \geq 0,$$

where $z = \begin{bmatrix} u \\ v \end{bmatrix}$, $y = A^T b, c = \mu e_{2n} + \begin{bmatrix} -y \\ y \end{bmatrix}$, and $H = \begin{bmatrix} A^T A & -A^T A \\ -A^T A & A^T A \end{bmatrix}$. Obviously, H is a positive semi-definite matrix, and for a given $z = \begin{bmatrix} u \\ v \end{bmatrix}$, the operations involving H can be performed economically,

$$Hz = H\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A^T A(u-v) \\ -A^T A(u-v) \end{bmatrix}$$

and

$$z^T H z = (u-v)^T A^T A(u-v) = \|A(u-v)\|_2^2.$$