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Lie symmetries, Conservation laws and Solutions for (4+1)-dimensional time fractional KP equation with variable coefficients in fluid mechanics

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Abstract. In recent years, high-dimensional fractional equations have gained prominence as a pivotal focus of interdisciplinary research spanning mathematical physics, fluid mechanics, and related fields. In this paper, we investigate a (4+1)-dimensional time-fractional Kadomtsev-Petviashvili (KP) equation with variable coefficients. We first derive the (4+1)-dimensional time-fractional KP equation with variable coefficients in the sense of the Riemann-Liouville fractional derivative using the semi-inverse and variational methods. The symmetries and conservation laws of this equation are analyzed through Lie symmetry analysis and a new conservation theorem, respectively. Finally, both exact and numerical solutions of the fractional-order equation are obtained using the Hirota bilinear method and the pseudo-spectral method. The effectiveness and reliability of the proposed approach are demonstrated by comparing the numerical solutions of the derived models with exact solutions in cases where such solutions are known.

AMS subject classifications: 22E47, 35G20, 35B10

Key words: Time fractional equation, Conservation laws, Hirota bilinear method, Pseudo-spectral method.

1 Introduction

In recent years, the research on high-dimensional integrability has gradually become a new hot topic [1, 2]. Many high-dimensional equations can describe extremely complex physical phenomena in nature. The study of high-dimensional nonlinear equations plays

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an important role in helping us understand some facts that cannot be understood by ordinary observation. In the previous study of nonlinear partial differential equation, many scholars realized the importance of high-dimensional nonlinear partial differential equation, and spent a lot of time to find the appropriate high-dimensional nonlinear partial differential equations [3–5].

KP equation was discovered in the study of nonlinear wave theory in weakly dispersive media by Kadomtsev and Petviashvili, physicists of the former Soviet Union. It possesses a broad physical background and significant applications in plasma physics, gas dynamics, and fluid mechanics. There are few studies on variable coefficient KP equation. The variable coefficient KP equation can describe the actual surface wave better than the constant coefficient KP equation. It can deal with the concrete situation of the surface wave entering the sea or ocean through the canyon when the width, depth and density change constantly. In recent years, with the high-dimensional nonlinear problems gradually become a hot topic, some (3+1)-dimensional KP equations [6–9] and (4+1)-dimensional KP equations have appeared. Fan et al. [10] first proposed a (4+1)-dimensional variable-coefficient KP equation in 2021, deriving lump solutions and interaction solutions including rogue waves and kink waves. Later, Zhu et al. [11] made some additions to the solutions of this equation. The equation has the form

$$\begin{aligned} f(t)u_x^2+f(t)uu_{xx}+g(t)u_{xxxx}+h_7(t)u_{ss}+h_6(t)u_{zz}+h_5(t)u_{yy} \\ +h_4(t)u_{xs}+h_3(t)u_{xz}+h_2(t)u_{xy}+h_1(t)u_{xx}+u_{xt}=0, \end{aligned} \tag{1.1}$$

where $u=u(x,y,s,z,t)$. $f(t)$ and $g(t)$ represent the nonlinearity and dispersion, respectively. $h_1(t)-h_4(t)$ stand for the perturbed effects. $h_5(t)-h_7(t)$ describe the disturbed wave velocities.

While the above equation is of integer order, fractional-order phenomena exist in natural systems as fundamental physical manifestations. At present, growing people pay attention to fractional equations, and the theory of fractional calculus is becoming more and more mature. Therefore, in this article, we try to extend the integer order (4+1) dimensional KP equation with variable coefficients to the fractional order form, and study the fractional equation. The time fractional form of the equation mentioned above has been derived for the first time using the semi-inverse method and the variational method [12]. This derivation has provided a more general significance to the equation. In our study, we focus on analyzing the symmetry, conservation laws, exact solutions, and numerical solutions of this equation.

Symmetry and conservation laws are very important for the study of partial differential equations. Recent advancements have been made in the investigation of non-classical Lie symmetries associated with partial differential equations. It is obvious that studies will be carried out on its application to fractional differential equations in the near future. Li's logarithmic method [13, 14] provides a robust framework for deriving analytical solutions to nonlinear partial differential equations. It was proposed by Markus Surface Li, a Norwegian mathematician. Gulsen [15] applied the technique that corresponds to non-classical symmetries to obtain new solutions to evolutionary-type equations. The nature of conservation comes from symmetry, and the conservation laws, as a generalization of physically conserved quantities such as energy conservation and momentum conservation, its important role in the development and research of nonlinear partial differential equations is mainly reflected in that the conservation laws can help solve and reduce nonlinear partial differential equations, construct special solutions of nonlinear partial differential equations, and help explain a large number of complex nonlinear physical phenomena described by nonlinear partial differential equations. For the construction of the conservation laws of integer-order PDEs, we are familiar with the method of Noether's theorem [16] and the new conservation theorem [17], and the new conservation theorem here is to construct the conservation laws based on the Lie point symmetries. For fractional partial differential equations, the conservation laws are usually constructed by using the extended Noether's theorem, but this method has its limitations [18, 19]. Until recently, some scholars proposed a method to construct the conservation laws of fractional partial differential equations by using the extended Noether's operator based on the new conservation theorem [20]. In our research, we propose a method that overcomes the limitations of previous approaches and no longer requires fractional partial differential equations to satisfy the fractional Lagrangian form. This method is applicable to constructing conservation laws for a wide range of fractional partial differential equations. Furthermore, the conservation laws for high-dimensional fractional order equations have not been extensively investigated. Therefore, in this paper, our focus is on constructing the conservation laws of the (4+1) dimensional time fractional KP equation with variable coefficients. By applying our method, we aim to provide new insights into the conservation properties of this specific equation.

Finding exact solutions for nonlinear partial differential equations is crucial in the field of mathematical physics as it allows us to gain a deeper understanding of the underlying nonlinear phenomena. The exact solutions of high-dimensional partial differential equations are discussed in the following article [21–27]. Zhao [28] uses the semi-inverse variational principle to obtain the soliton solutions of PDEs. In 2015, Ma [29] proposed a method to construct the Lump solutions of JM equation directly by using the Hirota bilinear method, and gave the theoretical proof and derivation, which pushed the research of Lump solutions to a new stage. Inspired by this, we employ Hirota bilinear method to solve the exact solutions of the (4+1) dimensional time fractional KP equation with variable coefficients. Hirota bilinear method is an effective method to construct exact solutions for many nonlinear partial differential equations, which plays an important role in nonlinear integrable systems. The key idea of this method is to convert the original nonlinear partial differential equations into bilinear form by means of some variable transformations and solve it by means of auxiliary functions. Some studies on this method can be found in the literature [30, 31]. In 2023, Yao [32] proposed Nucci's reduction method to obtain the exact solutions of the periodic Hunter–Suxon equation and got three separate families of vector fields. For numerical solutions, the finite difference method and finite element method [33] need a lot of computing costs and storage costs to deal with high-dimensional problems. The pseudo-spectral method [34, 35] offers an efficient alternative for numerically solving high-dimensional PDEs. Some scholars explored the application of this method in the numerical simulation of (3+1) dimensional seismic waves [36–38]. The advantage of this method is that it can obtain higher calculation accuracy in the case of large grid spacing, which will save a lot of calculation, reduce the burden of the computer, and is conducive to the numerical solution of high-dimensional partial differential equations. In this paper, we will apply the pseudo-spectral method to solve the (4+1)-dimensional time fractional KP equation with variable coefficients. And we obtained the satisfactory numerical solutions by the considered method.

The rest of this article is organized as follows. In Section 2, the (4+1)-dimensional time fractional KP equation with variable coefficients is derived by using the semi-inverse method and the variational approach. In Section 3, we employ Lie symmetry analysis to study the symmetry properties of the obtained time fractional KP equation with variable coefficients and utilize the new conservation theorem to construct the conservation laws associated with the equation [39–42]. In Section 4, the exact solutions of the time fractional KP equation with variable coefficients are given by using Hirota bilinear method. In Section 5, the numerical solutions of the time fractional KP equation with variable coefficients can be given by pseudo-spectral method. Finally, in Section 6, we present our conclusions based on the findings from our analysis of the exact and numerical solutions. These conclusions provide insights into the behavior and properties of the equation under study.

2 Derivation of the (4+1)-dimensional time fractional KP equation with variable coefficients

The (4+1)-dimensional KP equation with variable coefficients has the form

$$\begin{aligned} &f(t)u_x^2+f(t)uu_{xx}+g(t)u_{xxxx}+h_7(t)u_{ss}+h_6(t)u_{zz}+h_5(t)u_{yy}\\ &+h_4(t)u_{xs}+h_3(t)u_{xz}+h_2(t)u_{xy}+h_1(t)u_{xx}+u_{xt}=0, \end{aligned} \tag{2.1}$$

where $u=u(x,y,s,z,t)$. $f(t)$ and $g(t)$ represent the nonlinearity and dispersion, respectively. $h_1(t)$ to $h_4(t)$ stand for the perturbed effects. $h_5(t)$ to $h_7(t)$ describe the disturbed wave velocities.

Introducing a potential function $v(x,y,s,z,t)$ as $u(x,y,s,z,t)=v_x(x,y,s,z,t)$, we can rewrite the (4+1)-dimensional KP equation with variable coefficients as

$$\begin{aligned} &\frac{f(t)}{2}(u^2)_{xx}+g(t)v_{xxxxx}+h_7(t)v_{xss}+h_6(t)v_{xzz}+h_5(t)v_{xyy}+h_4(t)v_{xxs}\\ &+h_3(t)v_{xxz}+h_2(t)v_{xxy}+h_1(t)v_{xxx}+v_{xxt}=0, \end{aligned} \tag{2.2}$$

The functional of the potential Eq. (2.2) is

$$\begin{aligned} J(v)=&\int_R dx\int_Y dy\int_S ds\int_Z dz\int_T dt\{v[c_1\frac{f(t)}{2}(u^2)_{xx}+c_2g(t)v_{xxxxxx}\\ &+c_3h_7(t)v_{xss}+c_4h_6(t)v_{xzz}+c_5h_5(t)v_{xyy}+c_6h_4(t)v_{xxs}+c_7h_3(t)v_{xxz}\\ &+c_8h_2(t)v_{xxy}+c_9h_1(t)v_{xxx}+c_{10}v_{xxt}]\}, \end{aligned} \tag{2.3}$$

where $c_i(i=1,2,3,\cdots,10)$ is Lagrangian multipliers. Considering conditions that $(u^2)_{xx}$ is fixed function and $v_x|_R=v_x|_S=v_x|_Z=v_x|_Y=v_x|_T=0$, then integrating the above equation by parts. Using the variational of the above functional and the variational optimal condition $\delta J(v)=0$, we have

$$\begin{aligned} &c_1\frac{f(t)}{2}(u^2)_{xx}+2c_2g(t)v_{xxxxxx}+2c_3h_7(t)v_{xss}+2c_4h_6(t)v_{xzz}+2c_5h_5(t)v_{xyy}\\ &+2c_6h_4(t)v_{xxs}+2c_7h_3(t)v_{xxz}+2c_8h_2(t)v_{xxy}+2c_9h_1(t)v_{xxx}+2c_{10}v_{xxt}=0. \end{aligned} \tag{2.4}$$

We know Eq. (2.4) equals Eq. (2.2), so we obtain these constant coefficients: $c_1=1,c_j=\frac{1}{2}(j=2,3,4,\cdots,10)$. We can obtain the following Lagrangian form of Eq. (2.1) by substituting the values of $c_i(i=1,2,\cdots,10)$ into Eq. (2.3):

$$\begin{aligned} &L(v,v_x,v_y,v_s,v_z,v_t,v_{xx},v_{xy},v_{xs},v_{xz},v_{xxxx})\\ &=\frac{f(t)}{2}(u^2)_{xx}v-\frac{1}{2}g(t)v_xv_{xxxx}-\frac{1}{2}h_7(t)v_sv_{xs}-\frac{1}{2}h_6(t)v_zv_{xz}-\frac{1}{2}h_5(t)v_yv_{xy}\\ &-\frac{1}{2}h_4(t)v_sv_{xx}-\frac{1}{2}h_3(t)v_zv_{xx}-\frac{1}{2}h_2(t)v_yv_{xx}-\frac{1}{2}h_1(t)v_xv_{xx}-\frac{1}{2}v_tv_{xx}. \end{aligned} \tag{2.5}$$

Similarly, the Lagrangian form of the (4+1)-dimensional time fractional KP equation with variable coefficients can be obtained as

$$\begin{aligned} &F(v,v_x,v_y,v_s,v_z,D_t^{\alpha}v,v_{xx},v_{xy},v_{xs},v_{xz},v_{xxxx})\\ &=\frac{f(t)}{2}(u^2)_{xx}v-\frac{1}{2}g(t)v_xv_{xxxx}-\frac{1}{2}h_7(t)v_sv_{xs}-\frac{1}{2}h_6(t)v_zv_{xz}-\frac{1}{2}h_5(t)v_yv_{xy}\\ &-\frac{1}{2}h_4(t)v_sv_{xx}-\frac{1}{2}h_3(t)v_zv_{xx}-\frac{1}{2}h_2(t)v_yv_{xx}-\frac{1}{2}h_1(t)v_xv_{xx}-\frac{1}{2}D_t^{\alpha}vv_{xx}, \end{aligned} \tag{2.6}$$

where the D_t^{α} is Riemann-Liouville fractional derivative operator [43].

Consequently, the functional form of the equation with variable coefficients can be given as

$$J(v)=\int_R dx\int_Y dy\int_S ds\int_Z dz\int_T (dt)^{\alpha}F(v,v_x,v_y,v_s,v_z,D_t^{\alpha}v,v_{xx},v_{xy},v_{xs},v_{xz},v_{xxxx}), \tag{2.7}$$

where $\int_a^t(d\tau)^{\alpha}f(\tau)=\alpha\int_a^t d\tau(t-\tau)^{\alpha}f(\tau)$.

Integrating by parts for Eq. (2.7) with making use of the following relation [44] and variational optimal condition $\delta J(v)=0$:

$$\int_a^b (d\tau)^\alpha f(x) D_x^\alpha g(x) = \Gamma(1+\alpha)\left[g(x)f(x)|_a^b - \int_a^b (dx)^\alpha g(x) D_x^\alpha f(x)\right],$$
$$f(x), g(x) \in [a,b], \tag{2.8}$$

we can obtain the Euler-Lagrangian equation of the equation with variable coefficients

$$\begin{aligned}&\left(\frac{\partial F}{\partial v}\right)\cdot v + \left(\frac{\partial F}{\partial v_x}\right)\cdot v_x + \left(\frac{\partial F}{\partial v_y}\right)\cdot v_y + \left(\frac{\partial F}{\partial v_s}\right)\cdot v_s + \left(\frac{\partial F}{\partial v_z}\right)\cdot v_z + \left(\frac{\partial F}{\partial D_t^\alpha v}\right)\cdot D_t^\alpha v \\ &+ \left(\frac{\partial F}{\partial v_{xx}}\right)\cdot v_{xx} + \left(\frac{\partial F}{\partial v_{xy}}\right)\cdot v_{xy} + \left(\frac{\partial F}{\partial v_{xs}}\right)\cdot v_{xs} + \left(\frac{\partial F}{\partial v_{xz}}\right)\cdot v_{xz} + \left(\frac{\partial F}{\partial v_{xxxx}}\right)\cdot v_{xxxx} = 0.\end{aligned} \tag{2.9}$$

Substituting Eq. (2.6) into Eq. (2.9) and making use of the fractional potential function $D_x^\alpha v(x,y,s,z,t) = u(x,y,s,z,t)$, we can obtain the equation with variable coefficients

$$\begin{aligned}D_t^\alpha u_x + \frac{f(t)}{2}(u^2)_{xx} + g(t)u_{xxxx} + h_7(t)u_{ss} + h_6(t)u_{zz} + h_5(t)u_{yy} \\ + h_4(t)u_{xs} + h_3(t)u_{xz} + h_2(t)u_{xy} + h_1(t)u_{xx} = 0.\end{aligned} \tag{2.10}$$

Eq. (2.10) is the (4+1)-dimensional time fractional KP equation with variable coefficients. And $D_t^\alpha u$ can be defined as

$$D_t^\alpha u = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(x,s) ds, & n-1<\alpha<n, \\ \frac{\partial^n u}{\partial t^n}, & \alpha = n, \end{cases}$$

where $\Gamma(x)$ is Gamma function.

3 The symmetry analysis and conservation laws for the (4+1)-dimensional time fractional KP equation with variable coefficients

In this section, we study the Lie symmetry and conservation laws of the (4+1)-dimensional time fractional KP equation with variable coefficients. Some studies on Lie symmetry analysis and conservation laws of partial differential equations with variable coefficients may refer to these articles [45–47].

3.1 Lie symmetry analysis of the (4+1)-dimensional time fractional KP equation with variable coefficients

The (4+1)-dimensional time fractional KP equation with variable coefficients here has five variables x,y,s,z,t, the infinitesimal transformations are as follows.

Infinitesimal transformation of each variable:

$$
\begin{aligned}
x^* &= x + \epsilon\xi_1(x,y,s,z,t,u) + o(\epsilon^2),\\
y^* &= y + \epsilon\xi_2(x,y,s,z,t,u) + o(\epsilon^2),\\
s^* &= s + \epsilon\xi_3(x,y,s,z,t,u) + o(\epsilon^2),\\
z^* &= z + \epsilon\xi_4(x,y,s,z,t,u) + o(\epsilon^2),\\
t^* &= t + \epsilon\tau(x,y,s,z,t,u) + o(\epsilon^2),\\
u^* &= u + \epsilon\eta(x,y,s,z,t,u) + o(\epsilon^2),
\end{aligned}
\tag{3.1}
$$

where $\epsilon \ll 1$, $\xi_1,\xi_2,\xi_3,\xi_4,\tau,\eta$ are infinitesimal parameters. And the infinitesimal transformation of the partial derivatives of u with respect to different variables are

$$
\begin{aligned}
\frac{\partial u^*}{\partial x^*} &= \frac{\partial u}{\partial x} + \epsilon\eta^x(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial u^*}{\partial y^*} &= \frac{\partial u}{\partial y} + \epsilon\eta^y(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^\alpha u_x^*}{\partial t^{*\alpha}} &= \frac{\partial^\alpha u_x}{\partial t^\alpha} + \epsilon\eta^{\alpha,t}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial x^{*2}} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial y^{*2}} &= \frac{\partial^2 u}{\partial y^2} + \epsilon\eta^{yy}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial s^{*2}} &= \frac{\partial^2 u}{\partial s^2} + \epsilon\eta^{ss}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial z^{*2}} &= \frac{\partial^2 u}{\partial z^2} + \epsilon\eta^{zz}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial x^*\partial y^*} &= \frac{\partial^2 u}{\partial x\partial y} + \epsilon\eta^{xy}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial x^*\partial s^*} &= \frac{\partial^2 u}{\partial x\partial s} + \epsilon\eta^{xs}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial x^*\partial z^*} &= \frac{\partial^2 u}{\partial x\partial z} + \epsilon\eta^{xz}(x,y,s,z,t,u) + o(\epsilon^2),\\
\frac{\partial^4 u^*}{\partial x^{*4}} &= \frac{\partial^4 u}{\partial x^4} + \epsilon\eta^{xxxx}(x,y,s,z,t,u) + o(\epsilon^2),
\end{aligned}
\tag{3.2}
$$

where $\epsilon \ll 1$, $\eta^x,\eta^y,\eta^{xx},\eta^{yy},\eta^{ss},\eta^{zz},\eta^{xxxx},\eta^{xy},\eta^{xs},\eta^{xz},\eta^{\alpha,t}$ are extend infinitesimal parameters.

According to the reference [43], we can give $\eta^x,\eta^y,\eta^{xx},\eta^{xxxx},\eta^{xy},\eta^{xs},\eta^{xz},\eta^{\alpha,t}$ as

$$\begin{aligned}
\eta^x &= D_x(\eta)-u_xD_x(\xi_1)-u_yD_x(\xi_2)-u_sD_x(\xi_3)-u_zD_x(\xi_4)-u_tD_x(\tau),\\
\eta^{xx} &= D_x(\eta^x)-u_{xx}D_x(\xi_1)-u_{yx}D_x(\xi_2)-u_{sx}D_x(\xi_3)-u_{zx}D_x(\xi_4)-u_{tx}D_x(\tau),\\
\eta^{xy} &= D_y(\eta^x)-u_{xy}D_y(\xi_1)-u_{yy}D_y(\xi_2)-u_{sy}D_y(\xi_3)-u_{zy}D_y(\xi_4)-u_{ty}D_y(\tau),\\
\eta^{xs} &= D_s(\eta^x)-u_{xs}D_s(\xi_1)-u_{ys}D_s(\xi_2)-u_{ss}D_s(\xi_3)-u_{zs}D_s(\xi_4)-u_{ts}D_s(\tau),\\
\eta^{xz} &= D_z(\eta^x)-u_{xz}D_z(\xi_1)-u_{yz}D_z(\xi_2)-u_{sz}D_z(\xi_3)-u_{zz}D_z(\xi_4)-u_{tz}D_z(\tau),\\
\eta^{yy} &= D_y(\eta^y)-u_{xy}D_y(\xi_1)-u_{yy}D_y(\xi_2)-u_{sy}D_y(\xi_3)-u_{zy}D_y(\xi_4)-u_{ty}D_y(\tau),\\
\eta^{ss} &= D_s(\eta^s)-u_{xs}D_s(\xi_1)-u_{ys}D_s(\xi_2)-u_{ss}D_s(\xi_3)-u_{zs}D_s(\xi_4)-u_{ts}D_s(\tau),\\
\eta^{zz} &= D_z(\eta^z)-u_{xz}D_z(\xi_1)-u_{yz}D_z(\xi_2)-u_{sz}D_z(\xi_3)-u_{zz}D_z(\xi_4)-u_{tz}D_z(\tau),\\
\eta^{\alpha,t} &= D_t^\alpha(\eta^x)+\xi_1D_t^\alpha(u_{xx})-D_t^\alpha(\xi_1u_{xx})+\xi_2D_t^\alpha(u_{xy})-D_t^\alpha(\xi_2u_{xy})\\
&\quad+\xi_3D_t^\alpha(u_{xs})-D_t^\alpha(\xi_3u_{xs})+\xi_4D_t^\alpha(u_{xz})-D_t^\alpha(\xi_4u_{xz})+D_t^\alpha(D_t(\tau)u_x)\\
&\quad-D_t^{\alpha+1}(\tau u_x)+\tau D_t^{\alpha+1}(u_x),\\
\eta^{xxxx} &= D_x(\eta^{xxx})-u_{xxxx}D_x(\xi_1)-u_{yxxx}D_x(\xi_2)-u_{sxxx}D_x(\xi_3)-u_{zxxx}D_x(\xi_4)\\
&\quad-u_{txxx}D_x(\tau),
\end{aligned} \tag{3.3}$$

in which D_x,D_s,D_z and D_t are total derivative operators

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+u_{ty}\frac{\partial}{\partial u_y}+u_{ts}\frac{\partial}{\partial u_s}+u_{tz}\frac{\partial}{\partial u_z}+\cdots,\\
D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xy}\frac{\partial}{\partial u_y}+u_{xs}\frac{\partial}{\partial u_s}+u_{xz}\frac{\partial}{\partial u_z}+u_{xt}\frac{\partial}{\partial u_t}+\cdots,\\
D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{yx}\frac{\partial}{\partial u_x}+u_{ys}\frac{\partial}{\partial u_s}+u_{yz}\frac{\partial}{\partial u_z}+u_{yt}\frac{\partial}{\partial u_t}+\cdots,\\
D_s &= \frac{\partial}{\partial s}+u_s\frac{\partial}{\partial u}+u_{ss}\frac{\partial}{\partial u_s}+u_{sx}\frac{\partial}{\partial u_x}+u_{sy}\frac{\partial}{\partial u_y}+u_{sz}\frac{\partial}{\partial u_z}+u_{st}\frac{\partial}{\partial u_t}+\cdots,\\
D_z &= \frac{\partial}{\partial z}+u_z\frac{\partial}{\partial u}+u_{zz}\frac{\partial}{\partial u_z}+u_{zx}\frac{\partial}{\partial u_x}+u_{zy}\frac{\partial}{\partial u_y}+u_{zs}\frac{\partial}{\partial u_s}+u_{zt}\frac{\partial}{\partial u_t}+\cdots.
\end{aligned} \tag{3.4}$$

The infinitesimal generators X has the following form:

$$\begin{aligned}
X &= \xi_1(x,y,s,z,t,u)\frac{\partial}{\partial x}+\xi_2(x,y,s,z,t,u)\frac{\partial}{\partial y}+\xi_3(x,y,s,z,t,u)\frac{\partial}{\partial s}\\
&\quad+\xi_4(x,y,s,z,t,u)\frac{\partial}{\partial z}+\tau(x,y,s,z,t,u)\frac{\partial}{\partial t}+\eta(x,y,s,z,t,u)\frac{\partial}{\partial u}.
\end{aligned} \tag{3.5}$$

Infinitesimal invariant criterion obtained under infinitesimal transformation is

$$pr^{(\alpha)}X(\Delta)|_{\Delta=0}=0, \tag{3.6}$$

where

$$\begin{aligned}
\Delta &= D_t^\alpha u_x+\frac{f(t)}{2}(u^2)_{xx}+g(t)u_{xxxx}+h_7(t)u_{ss}+h_6(t)u_{zz}\\
&\quad+h_5(t)u_{yy}+h_4(t)u_{xs}+h_3(t)u_{xz}+h_2(t)u_{xy}+h_1(t)u_{xx}=0,
\end{aligned} \tag{3.7}$$

and the prolongation operator $pr^{(\alpha)}X$ is

$$pr^{(\alpha)}X=X+\eta^{\alpha,t}\frac{\partial}{D_t^{\alpha}u_x}+\eta^{x}\frac{\partial}{\partial u_x}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xy}\frac{\partial}{\partial u_{xy}}+\eta^{xs}\frac{\partial}{\partial u_{xs}}$$
$$+\eta^{xz}\frac{\partial}{\partial u_{xz}}+\eta^{yy}\frac{\partial}{\partial u_{yy}}+\eta^{ss}\frac{\partial}{\partial u_{ss}}+\eta^{zz}\frac{\partial}{\partial u_{zz}}+\eta^{xxxx}\frac{\partial}{\partial u_{xxxx}}. \tag{3.8}$$

The structure of the fractional derivative remains invariant under the transformations given by Eqs.(3.1) and (3.2). It is worth noting that the lower limit of the integral in Eq. (2.10) is fixed and should also remain invariant under these transformations. The invariant condition yields $\tau(x,y,s,z,t,u)|_{t=0}=0$.

Besides, the generalized chain rule and generalized Leibnitz rule are defined as [40,41]:

$$\frac{d^m g(y(t))}{dt^m}=\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-y(t)]^r\frac{d^m}{dt^m}\left[(y(t))^{k-r}\right]\frac{d^k g(y)}{dy^k}, \tag{3.9}$$

$$D_t^{\alpha}(f(t)g(t))=\sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^{\alpha-n}(f(t))D_t^{n}(g(t)),\alpha>0, \tag{3.10}$$

where $\binom{\alpha}{n}=\frac{-1^{(n-1)}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$.

Now, making use of Eqs.(3.3)-(3.9) and Eq. (3.10) with $f(t)=1$, so we can get the specific expression of the extended infinitesimal. Take $\eta^x,\eta^y,\eta^{xx},\eta^{yy},\eta^{xy},\eta^{\alpha,t}$ for example to demonstrate

$$\begin{aligned}
\eta^x=&\eta_x+u_x\eta_u-u_x^2\xi_{1u}-u_y\xi_{2x}-u_yu_x\xi_{2u}-u_s\xi_{3x}\\
&-u_su_x\xi_{3u}-u_t\tau_x-u_tu_x\tau_u-u_z\xi_{4x}-u_zu_x\xi_{4u},\\
\eta^y=&\eta_y+u_y\eta_u-u_x\xi_{1y}-u_xu_y\xi_{1u}-u_y\xi_{2y}-u_y^2\xi_{2u}-u_s\xi_{3y}\\
&-u_su_{yx_2}\xi_{3u}-u_t\tau_y-u_tu_y\tau_u-u_z\xi_{4y}-u_zu_y\xi_{4u},\\
\eta^{xx}=&\eta_{xx}+u_{xx}\eta_u+2u_x\eta_{ux}-2u_{xx}\xi_{1x}-u_x\xi_{1xx}-3u_xu_{xx}\xi_{1u}-2u_x^2\xi_{1ux}\\
&-2u_{yx}\xi_{2x}-u_y\xi_{2xx}-u_{xx}u_y\xi_{2u}-2u_xu_{yx}\xi_{2u}-2u_xu_y\xi_{2ux}-2u_{sx}\xi_{3x}\\
&-u_s\xi_{3xx}-2u_{sx}u_x\xi_{3u}-u_su_{xx}\xi_{3u}-2u_su_x\xi_{3ux}-2u_{zx}\xi_{4x}-u_z\xi_{4xx}\\
&-2u_{zx}u_x\xi_{4u}-u_zu_{xx}\xi_{4u}-2u_zu_x\xi_{4ux}-2u_{tx}\tau_x-u_t\tau_{xx}-2u_{tx}u_x\tau_u\\
&-u_tu_{xx}\tau_u-2u_tu_x\tau_{ux}+u_x^2\eta_{uu}-u_x^3\xi_{1uu}-u_x^2u_y\xi_{2uu}-u_x^2u_s\xi_{3uu}\\
&-u_x^2u_z\xi_{4uu}-u_x^2u_t\tau_{uu},\\
\eta^{yy}=&\eta_{yy}+u_{yy}\eta_u+2u_y\eta_{uy}-2u_{xy}\xi_{1y}-u_x\xi_{1yy}-2u_{xy}u_y\xi_{1u}-u_xu_{yy}\xi_{1u}\\
&-2u_xu_y\xi_{1uy}-2u_{yy}\xi_{2y}-u_y\xi_{2yy}-3u_yu_{yy}\xi_{2u}-2u_y^2\xi_{2uy}-2u_{sy}\xi_{3y}-u_s\xi_{3yy}\\
&-2u_{sy}u_y\xi_{3u}-u_su_{yy}\xi_{3u}-2u_su_y\xi_{3uy}-2u_{zy}\xi_{4y}-u_z\xi_{4yy}-2u_{zy}u_y\xi_{4u}\\
&-u_zu_{yy}\xi_{4u}-2u_zu_y\xi_{4uy}-2u_{ty}\tau_y-u_t\tau_{yy}-2u_{ty}u_y\tau_u-u_tu_{yy}\tau_u-2u_tu_y\tau_{uy}\\
&+u_y^2\eta_{uu}-u_xu_y^2\xi_{1uu}-u_y^3\xi_{2uu}-u_su_y^2\xi_{3uu}-u_zu_y^2\xi_{4uu}-u_tu_y^2\tau_{uu},
\end{aligned}$$

$$
\begin{aligned}
\eta^{xy} = & \eta_{xy} + u_{xy}\eta_u + u_x\eta_{uy} - u_{xy}\xi_{1x} - u_x\xi_{1xy} - 2u_xu_{xy}\xi_{1u} - u_x^2\xi_{1uy} - u_{yy}\xi_{2x} \\
& - u_y\xi_{2xy} - u_{xy}u_y\xi_{2u} - u_xu_{yy}\xi_{2u} - u_xu_y\xi_{2uy} - u_{sy}\xi_{3x} - u_s\xi_{3y} - u_{sy}u_x\xi_{3u} \\
& - u_su_{xy}\xi_{3u} - u_su_x\xi_{3uy} - u_{zy}\xi_{4x} - u_z\xi_{4xy} - u_{zy}u_x\xi_{4u} - u_zu_{xy}\xi_{4u} - u_zu_x\xi_{4uy} \\
& - u_{ty}\tau_x - u_t\tau_{xy} - u_{ty}u_x\tau_u - u_tu_{xy}\tau_u - u_tu_x\tau_{uy} + u_y\eta_{xu} + u_xu_y\eta_{uu} \\
& - u_xu_y\xi_{1u} - u_x^2u_y\xi_{1uu} - u_y^2\xi_{2xu} - u_xu_y^2\xi_{2uu} - u_su_y\xi_{3xu} - u_su_xu_y\xi_{3uu} \\
& - u_zu_y\xi_{4xu} - u_zu_xu_y\xi_{4uu} - u_tu_y\tau_{xu} - u_tu_xu_y\tau_{uu} - u_{xy}\xi_{1y} - u_{xy}u_y\xi_{1u} \\
& - u_{yy}\xi_{2y} - u_{yy}u_y\xi_{2u} - u_{sy}\xi_{3y} - u_{sy}u_y\xi_{3u} - u_{zy}\xi_{4y} - u_{zy}u_y\xi_{4u} - u_{ty}\tau_y \\
& - u_{ty}u_y\tau_u, \\
\eta^{\alpha,t} = & \partial_t^\alpha(\eta^x) + [(\eta^x)_u + \alpha D_t(\tau)]\partial_t^\alpha u - u\partial_t^\alpha(\eta^x)_u + \mu \\
& + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^n(\eta^x)_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]\partial_t^{\alpha-n}u - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi_1)\partial_t^{\alpha-n}(u_{xx}) \\
& - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi_2)\partial_t^{\alpha-n}(u_{xy}) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi_3)\partial_t^{\alpha-n}(u_{xs}) \\
& - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi_4)\partial_t^{\alpha-n}(u_{xz}),
\end{aligned}
$$

where

$$
\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k}.
$$

Substituting Eqs.(3.6),(3.5),(3.7) into Eq. (3.8), we have

$$
\begin{aligned}
& \tau f'(t)u_x^2 + \tau f'(t)uu_{xx} + \tau g'(t)u_{xxxx} + \tau h_7'(t)u_{ss} + \tau h_6'(t)u_{zz} + \tau h_5'(t)u_{yy} \\
& + \tau h_4'(t)u_{xs} + \tau h_3'(t)u_{xz} + \tau h_2'(t)u_{xy} + \tau h_1'(t)u_{xx} + \eta f(t)u_{xx} + \eta^{\alpha,t} \\
& + 2\eta^x f(t)u_x + \eta^{xx}[f(t)u + h_1(t)] + h_2(t)\eta^{xy} + h_4(t)\eta^{xs} + h_3(t)\eta^{xz} \\
& + h_5(t)\eta^{yy} + h_7(t)\eta^{ss} + h_6(t)\eta^{zz} + g(t)\eta^{xxxx} = 0.
\end{aligned} \tag{3.11}
$$

By substituting the specific expression of the extended infinitesimal parameters into Eq. (3.11), we can derive the determining equations by equating the coefficients of the partial derivatives of u of different orders to zero. This step allows us to obtain a set of equations that determine the form of the solution u. Then simplifying these equations, we have

$$
\begin{aligned}
& \tau f'(t) + 2f(t)(\eta_u - \xi_{1x}) = 0, \\
& \tau h_1'(t) + \eta f(t) + h_1(t)(\eta_u - 2\xi_{1x}) = 0, \\
& \tau h_2'(t) + h_2(t)(\eta_u - \xi_{1x}) = 0, \\
& \tau h_3'(t) + h_3(t)(\eta_u - \xi_{1x}) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\tau h_4^{'}(t)+h_4(t)(\eta_u-\xi_{1x})=0,\\
&h_5(t)(\eta_u-2\xi_{2y})-h_2(t)\xi_{2y}+\tau h_5'(t)=0,\\
&h_6(t)(\eta_u-2\xi_{4z})-h_3(t)\xi_{4z}+\tau h_6'(t)=0,\\
&h_7(t)(\eta_u-2\xi_{3s})-h_4(t)\xi_{3s}+\tau h_7'(t)=0,\\
&g(t)(\eta_u-4\xi_{1x})+\tau g'(t)=0,\\
&\xi_{1u}=\xi_{1y}=\xi_{1s}=\xi_{1z}=\xi_{1t}=0,\\
&\xi_{2u}=\xi_{2x}=\xi_{2s}=\xi_{2z}=\xi_{2t}=0,\\
&\xi_{3u}=\xi_{3x}=\xi_{3y}=\xi_{3z}=\xi_{3t}=0,\\
&\xi_{4u}=\xi_{4x}=\xi_{4y}=\xi_{4s}=\xi_{4t}=0,\\
&\tau_u=\tau_x=\tau_y=\tau_s=\tau_z=0,\\
&\eta_u=\eta_x=\eta_{yy}=\eta_{ss}=\eta_{zz}=0,\\
&\binom{\alpha}{n}\partial_t^n(\eta^x)_u-\binom{\alpha}{n+1}D_t^{n+1}(\tau)=0.
\end{aligned}
$$

Solving the above equations, we can get a set of nontrivial solutions:

$$
\begin{aligned}
&\eta=\lambda(y,s,z),\xi_1=A_1x+d_1,\xi_2=A_2y+d_2,\xi_3=A_3s+d_3,\\
&\xi_4=A_4z+d_4,\tau=\frac{2A_1f(t)}{f'(t)}.
\end{aligned}
\tag{3.12}
$$

where $A_i,d_j(i=1,2,3,4,j=1,2,3,4)$ are arbitrary constants, $\lambda(y,s,z)$ satisfies $\lambda_{yy}=\lambda_{ss}=\lambda_{zz}=0$ and $f(t),g(t),h_1(t)-h_7(t)$ satisfy

$$
\begin{aligned}
&\tau f^{'}(t)-2A_1f(t)=0,\\
&\tau h_1^{'}(t)-2A_1h_1(t)=0,\\
&\tau h_2^{'}(t)-A_1h_2(t)=0,\\
&\tau h_3^{'}(t)-A_1h_3(t)=0,\\
&\tau h_4^{'}(t)-A_1h_4(t)=0,\\
&\tau h_5'(t)-2A_2h_5(t)-A_2h_2(t)=0,\\
&\tau h_6'(t)-2A_4h_6(t)-A_4h_3(t)=0,\\
&\tau h_7'(t)-2A_3h_7(t)-A_3h_4(t)=0,\\
&\tau g'(t)-4A_1g(t)=0.
\end{aligned}
\tag{3.13}
$$

According to Eq. (3.5), the corresponding infinitesimal generator can be written as follows:

$$
\begin{aligned}
X=&(A_1x+d_1)\frac{\partial}{\partial x}+(A_2y+d_2)\frac{\partial}{\partial y}+(A_3s+c_3)\frac{\partial}{\partial s}\\
&+(A_4z+d_4)\frac{\partial}{\partial z}+\left(\frac{2A_1f(t)}{f'(t)}\right)\frac{\partial}{\partial t}+\lambda(y,s,z)\frac{\partial}{\partial u},
\end{aligned}
\tag{3.14}
$$

thus, we can get the corresponding Lie algebra that can be spanned by the following six vector fields:

$$X_1=\frac{\partial}{\partial x}, X_2=\frac{\partial}{\partial y}, X_3=\frac{\partial}{\partial s}, X_4=\frac{\partial}{\partial z}, X_5=\frac{2A_1 f(t)}{f'(t)}\frac{\partial}{\partial t}+\lambda(y,s,z)\frac{\partial}{\partial u},$$
$$X_6=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+s\frac{\partial}{\partial s}+z\frac{\partial}{\partial z}. \tag{3.15}$$

3.2 Conservation laws of the (4+1)-dimensional time fractional KP equation with variable coefficients

In this section, the conservation laws of (4+1)-dimensional time fractional KP equation with variable coefficients can be constructed by the new conservation laws theorem.

Definition 3.1. A conservation laws for Eq. (2.10) can be expressed by the following conservation equation:

$$D_t(C^t)+D_x(C^x)+D_y(C^y)+D_s(C^s)+D_z(C^z)|_{(10)}=0, \tag{3.16}$$

where $C=(C^t,C^x,C^y,C^s,C^z)$ is conserved vector. According to the Noether operators, we can obtain the components C^t,C^x,C^y,C^s and C^z of conserved vector C as

$$C^t=\tau\mathcal{L}+\sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W)D_t^k\left(\frac{\partial\mathcal{L}}{\partial(D_t^\alpha u)}\right)-(-1)^n J\left(W, D_t^n\left(\frac{\partial\mathcal{L}}{\partial(D_t^\alpha u)}\right)\right), \tag{3.17}$$

and C^i(i stands for x,y,s,z) can be defined as

$$C^i=\xi^i+W_\beta\left[\frac{\partial\mathcal{L}}{\partial u_i^\beta}-D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^\beta}\right)+D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\beta}\right)-\cdots\right]$$
$$+D_j(W_\beta)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\beta}-D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\beta}\right)+\cdots\right]+D_jD_k(W_\beta)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\beta}-\cdots\right]+\cdots, \tag{3.18}$$

where $n=[\alpha]+1$, $W=\eta-\xi_1u_x-\xi_2u_y-\xi_3u_s-\xi_4u_z-\tau u_t$ is Lie characteristic function of $X=\xi_1\partial_x+\xi_2\partial_y+\xi_3\partial_s+\xi_4\partial_z+\tau\partial_t+\eta\partial_u$, and J is defined as

$$J(f,g)=\frac{1}{\Gamma(n-\beta)}\int_0^t\int_t^T\frac{f(x,s)g(x,r)}{(r-s)^{\beta+1-n}}drds. \tag{3.19}$$

Now, based on Lie point symmetry, we start to construct the conservation laws of Eq. (2.10). A formal Lagrangian for Eq. (2.10) is given in the form

$$\mathcal{L}=q(x,y,s,z,t)(D_t^\alpha u_x+\frac{f(t)}{2}(u^2)_{xx}+g(t)u_{xxxx}+h_7(t)u_{ss}+h_6(t)u_{zz}+h_5(t)u_{yy}$$
$$+h_4(t)u_{xs}+h_3(t)u_{xz}+h_2(t)u_{xy}+h_1(t)u_{xx}), \tag{3.20}$$

where $q(x,y,s,z,t)$ is a new dependent variable. Considering the case where the variable q is constant, we integrate the above equation using the Agrawal fractional variational method. This allows us to determine the Euler-Lagrange operator [42] with respect to u. By applying this operator to the Lagrangian, we can obtain the corresponding Euler-Lagrange equations that govern the behavior of u in the given system

$$\begin{aligned}\frac{\delta}{\delta u}=&\frac{\partial}{\partial u}+(D_t^\alpha)^*D_x\frac{\partial}{\partial D_t^\alpha u_x}-D_x\frac{\partial}{\partial u_x}+D_{xx}\frac{\partial}{\partial u_{xx}}+D_{xy}\frac{\partial}{\partial u_{xy}}+D_{xs}\frac{\partial}{\partial u_{xs}}\\&+D_{xz}\frac{\partial}{\partial u_{xz}}+D_{xxxx}\frac{\partial}{\partial u_{xxxx}}+D_{yy}\frac{\partial}{\partial u_{yy}}+D_{ss}\frac{\partial}{\partial u_{ss}}+D_{zz}\frac{\partial}{\partial u_{zz}},\end{aligned}\tag{3.21}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α

$$(D_t^\alpha)^*=(-1)^nI_T^{n-\alpha}(D_t^n)={}_t^CD_T^\alpha,$$

in which, the time-fractional integral with order $n-\alpha$ can be given by ([38])

$$I_T^{n-\alpha}f(t,x)=\frac{1}{\Gamma(n-\alpha)}\int_t^T\frac{f(\tau,x)}{(\tau-t)^{1+\alpha-n}}d\tau,\qquad n=[\alpha]+1.\tag{3.22}$$

The adjoint equation of Eq. (2.10) can be given as

$$F^*=\frac{\delta\mathcal{L}}{\delta u}=0.\tag{3.23}$$

Expanding the above formula to obtain

$$\begin{aligned}F^*=&(D_t^\alpha)^*q_x-f(t)u_{xx}q-2f(t)u_xq_x+h_1(t)q_{xx}+h_2(t)q_{xy}+h_3(t)q_{xz}\\&+h_4(t)q_{xs}+h_5(t)q_{yy}+h_6(t)q_{zz}+h_7(t)q_{ss}+g(t)q_{xxxx}.\end{aligned}\tag{3.24}$$

According to Eq. (3.12), we get the Lie characteristic function

$$\begin{aligned}&W_1=-u_x,W_2=-u_y,W_3=-u_s,W_4=-u_z,W_5=\lambda(y,s,z)-\frac{2f(t)}{f'(t)}u_t,\\&W_6=-xu_x-yu_y-su_s-zu_z.\end{aligned}\tag{3.25}$$

Taking an example of W_6 to obtain the conservation laws for Eq. (2.10). By definition 1, substituting W_6 into Eq. (3.17) and Eq. (3.18), the conserved components with respect to x,y,s,z,t of conserved vector C can be got as

$$\begin{aligned}C^t=&D_t^{\alpha-1}(W_6)\frac{\partial\mathcal{L}}{\partial(D_t^\alpha u_x)}+J\left(W_6,D_t\frac{\partial\mathcal{L}}{\partial D_t^\alpha u_x}\right)\\=&qD_t^{\alpha-1}(-xu_x-yu_y-su_s-zu_z)+J[(-xu_x-yu_y-su_s-zu_z),q_t],\end{aligned}\tag{3.26}$$

$$\begin{aligned}
C^x =& W_6\left(\frac{\partial\mathcal{L}}{\partial u_x}-D_x\frac{\partial\mathcal{L}}{\partial u_{xx}}-D_y\frac{\partial\mathcal{L}}{\partial u_{xy}}-D_s\frac{\partial\mathcal{L}}{\partial u_{xs}}-D_z\frac{\partial\mathcal{L}}{\partial u_{xz}}-D_x^3\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\right)\\
&+D_x(W_6)\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}+D_xD_x\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\right)+D_y(W_6)\frac{\partial\mathcal{L}}{\partial u_{xy}}+D_s(W_6)\frac{\partial\mathcal{L}}{\partial u_{xs}}\\
&+D_z(W_6)\frac{\partial\mathcal{L}}{\partial u_{xz}}+D_x^3(W_6)\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\\
=&(-xu_x-yu_y-su_s-zu_z)(2f(t)q-f(t)u_xq-f(t)uq_x-h_1(t)q_x\\
&\quad -h_2(t)q_y-h_4(t)q_s-h_3(t)q_z-g(t)q_{xxxx})\\
&+(-u_x-xu_{xx}-yu_{yx}-su_{sx}-zu_{zx})(f(t)uq+h_1(t)q+g(t)q_{xx})\\
&+(-xu_{xy}-u_y-yu_{yy}-su_{sy}-zu_{zy})(h_2(t)q)\\
&+(-xu_{xs}-yu_{ys}-u_s-su_{ss}-zu_{zs})(h_4(t)q)\\
&+(-xu_{xz}-yu_{yz}-su_{sz}-u_z-zu_{zz})(h_3(t)q)\\
&+(-3u_{xxx}-xu_{xxxx}-yu_{yxxx}-su_{sxxx}-zu_{zxxx})(g(t)q),
\end{aligned}\tag{3.27}$$

$$\begin{aligned}
C^y =& W_6\left(-D_x\frac{\partial\mathcal{L}}{\partial u_{xy}}-D_y\frac{\partial\mathcal{L}}{\partial u_{yy}}\right)+D_x(W_6)\left(\frac{\partial\mathcal{L}}{\partial u_{xy}}\right)+D_y(W_6)\frac{\partial\mathcal{L}}{\partial u_{yy}}\\
=&(xu_x+yu_y+su_s+zu_z)(h_1(t)q_x+h_5(t)q_y)\\
&-(u_x+xu_{xx}+yu_{yx}+su_{sx}+zu_{zx})(h_2(t)q)\\
&-(xu_{xy}+u_y+yu_{yy}+su_{sy}+zu_{zy})(h_5(t)q),
\end{aligned}\tag{3.28}$$

$$\begin{aligned}
C^s =& W_6\left(-D_x\frac{\partial\mathcal{L}}{\partial u_{xs}}-D_s\frac{\partial\mathcal{L}}{\partial u_{ss}}\right)+D_x(W_6)\left(\frac{\partial\mathcal{L}}{\partial u_{xs}}\right)+D_s(W_6)\frac{\partial\mathcal{L}}{\partial u_{ss}}\\
=&(xu_x+yu_y+su_s+zu_z)(h_4(t)q_x+h_7(t)q_s)\\
&-(u_x+xu_{xx}+yu_{yx}+su_{sx}+zu_{zx})(h_4(t)q)\\
&-(xu_{xs}+yu_{ys}+u_s+su_{ss}+zu_{zs})(h_7(t)q),
\end{aligned}\tag{3.29}$$

$$\begin{aligned}
C^z =& W_6\left(-D_x\frac{\partial\mathcal{L}}{\partial u_{xz}}-D_z\frac{\partial\mathcal{L}}{\partial u_{zz}}\right)+D_x(W_6)\left(\frac{\partial\mathcal{L}}{\partial u_{xz}}\right)+D_z(W_6)\frac{\partial\mathcal{L}}{\partial u_{zz}}\\
=&(xu_x+yu_y+su_s+zu_z)(h_3(t)q_x+h_6(t)q_z)\\
&-(u_x+xu_{xx}+yu_{yx}+su_{sx}+zu_{zx})(h_3(t)q)\\
&-(xu_{xz}+yu_{yz}+su_{sz}+u_z+zu_{zz})(h_6(t)q)
\end{aligned}\tag{3.30}$$

4 Exact solutions for the (4+1)-dimensional time fractional KP equation with variable coefficients

Definition 4.1. Suppose the functions $f(x,y,z,t)$ and $g(x,y,z,t)$ are differentiable, Hirota bilinear derivative operator can be written as

$$D_x^\alpha D_y^\beta D_z^\gamma D_t^\eta(f\cdot g)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^\alpha\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^\beta\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^\gamma\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^\eta.$$

$$f(x,y,z,t)g(x',y',z',t')|_{x=x',y=y',z=z',t=t'}. \tag{4.1}$$

where α,β,γ,η are non-negative integer. With respect to the expansion of the bilinear operator m_{th} in the above formula, the result is a binomial expression of the following formula:

$$D_x^m(f\cdot g)=\sum_{r=0}^{m}(-1)^r\binom{m}{r}\partial_x^r\partial_{x'}^{m-r}=\sum_{r=0}^{m}(-1)^r\binom{m}{r}\partial_x^{m-r}\partial_{x'}^r, \tag{4.2}$$

where the binomial coefficient: $\binom{m}{r}=\frac{m!}{r!(m-r)!},0\le r\le m$. So we have a compact form of the bilinear derivative operator

$$D_x^m(f\cdot g)=\sum_{r=0}^{m}(-1)^r\binom{m}{r}f_{(m-r)x}\cdot g_{rx}. \tag{4.3}$$

Common Hirota bilinear derivative operators are

$$\begin{aligned}
&D_x^1(f\cdot g)=f_xg-fg_x,\\
&D_x^2(f\cdot g)=f_{xx}g-2f_xg_x+fg_{xx},\\
&D_x^3(f\cdot g)=f_{xxx}g-3f_{xx}g_x+3f_xg_{xx}-fg_{xxx},\\
&D_x^4(f\cdot g)=f_{xxxx}g-4f_{xxx}g_x+6f_{xx}g_{xx}-4f_xg_{xxx}+fg_{xxxx}.
\end{aligned} \tag{4.4}$$

We introduce the fractional transform

$$T=\frac{mt^\alpha}{\Gamma(1+\alpha)}. \tag{4.5}$$

Using the Eq. (4.5) with $m=1$, we can write the Eq. (2.10) as

$$\begin{aligned}
&u_{xT}+\frac{f(t)}{2}(u^2)_{xx}+g(t)u_{xxxx}+h_7(t)u_{ss}+h_6(t)u_{zz}+h_5(t)u_{yy}\\
&+h_4(t)u_{xs}+h_3(t)u_{xz}+h_2(t)u_{xy}+h_1(t)u_{xx}=0.
\end{aligned} \tag{4.6}$$

Considering the transformation

$$u(x,y,s,z,T)=Rln(f)_{xx}, \tag{4.7}$$

where $f(x,y,s,z,T)$ is an auxiliary function, and substituting Eq. (4.7) into Eq. (4.6), we can get $R=\frac{12g(t)}{f(t)}$. Under the specified transformation Eq. (4.7), the Hirota's bilinear form of Eq. (4.6) can be obtained as

$$\begin{aligned}
&(g(t)D_x^4+h_7(t)D_s^2+h_6(t)D_s^2+h_5(t)D_y^2+h_4(t)D_xD_s+h_3(t)D_sD_z\\
&+h_2(t)D_xD_(y)+h_1(t)D_x^2+D_xD_T)f\cdot f=0.
\end{aligned} \tag{4.8}$$

4.1 Single soliton solutions and double soliton solutions

To get the single soliton solutions of Eq. (4.6), we assume $f(x,y,s,z,T)$ as the following form:

$$f(x,y,s,z,T)=1+e^{\theta(x,y,s,z,T)}, \tag{4.9}$$

where $\theta(x,y,s,z,T)=kx+py+qs+rz+wT+c$, k,p,q,r,c are constants and w is dispersion relation to be determined. Substituting Eq. (4.9) into Eq. (4.8), the dispersion relation can be obtained as

$$w=-k^3g(t)-qh_4(t)-rh_3(t)-ph_2(t)-kh_1(t)-\frac{q^2h_7(t)+r^2h_2(t)+p^2h_5(t)}{k}, \tag{4.10}$$

A direct substitution of Eq. (4.10) into Eq. (4.9), then substituting Eq. (4.9) into Eq. (4.7) with $R=\frac{12g(t)}{f(t)}$, the single soliton solutions can be obtained as

$$u(x,y,s,z,t)=\frac{3g(t)}{f(t)}k^2sech^2\left(\frac{e^{kx+py+qs+rz+w\frac{t^\alpha}{\Gamma(1+\alpha)}+c}}{2}\right). \tag{4.11}$$

For double soliton solution, we assume $f(x,y,s,z,T)$ as the form

$$f(x,y,s,z,T)=1+e^{\theta_1(x,y,s,z,T)}+e^{\theta_2(x,y,s,z,T)}+h_{12}e^{\theta_1(x,y,s,z,T)+\theta_2(x,y,s,z,T)}, \tag{4.12}$$

where $\theta_1(x,y,s,z,T)=k_1x+p_1y+q_1s+r_1z+w_1T+c_1,\theta_2(x,y,s,z,T)=k_2x+p_2y+q_2s+r_2z+w_2T+c_2$. According to Eq. (4.10), we have the dispersion relations

$$\begin{aligned} w_1&=-k_1^3g(t)-q_1h_4(t)-r_1h_3(t)-p_1h_2(t)-k_1h_1(t)-\frac{q_1^2h_7(t)+r_1^2h_2(t)+p_1^2h_5(t)}{k},\\ w_2&=-k_2^3g(t)-q_2h_4(t)-r_2h_3(t)-p_2h_2(t)-k_2h_1(t)-\frac{q_2^2h_7(t)+r_2^2h_2(t)+p_2^2h_5(t)}{k}. \end{aligned} \tag{4.13}$$

Substituting Eq. (4.13) into Eq. (4.12), then substituting Eq. (4.12) into Eq. (4.8), we can get the interaction coefficient

$$h_{12}=\frac{M}{N}, \tag{4.14}$$

where

$$M=3k_1^2k_2^2(k_1-k_2)^2g(t)-[(k_1q_2-k_2q_1)^2h_7(t)+(k_1r_2-k_2r_1)^2h_6(t)+(k_1p_2-k_2p_1)^2h_5(t)],$$
$$N=3k_1^2k_2^2(k_1+k_2)^2g(t)-[(k_1q_2-k_2q_1)^2h_7(t)+(k_1r_2-k_2r_1)^2h_6(t)+(k_1p_2-k_2p_1)^2h_5(t)].$$

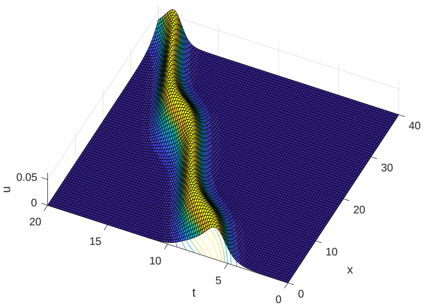
So, according to Eqs.(4.7),(4.12)-(4.14), the double soliton solutions can be obtained as the form

$$\begin{aligned} u(x,y,s,z,t)=&\frac{12g(t)}{f(t)}[k_1^2e^{\theta_1(x,y,s,z,t)}+h_{12}(k_2^2e^{\theta_1(x,y,s,z,t)}+k_1^2e^{\theta_2(x,y,s,z,t)})\\ &e^{\theta_1(x,y,s,z,t)+\theta_2(x,y,s,z,t)}+k_2^2e^{\theta_2(x,y,s,z,t)}+((k_1-k_2)^2+h_{12}(k_1+k_2)^2)\\ &e^{\theta_1(x,y,s,z,t)+\theta_2(x,y,s,z,t)}]/(1+e^{\theta_1(x,y,s,z,t)}+e^{\theta_2(x,y,s,z,t)}\\ &+h_{12}e^{\theta_1(x,y,s,z,t)+\theta_2(x,y,s,z,t)})^2, \end{aligned} \tag{4.15}$$

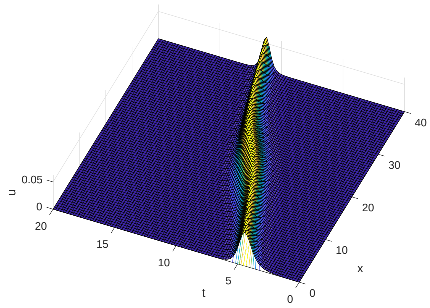
where $\theta_1(x,y,s,z,t),\theta_2(x,y,s,z,t)$ can be written as

$$\theta_i(x,y,s,z,t)=k_i x+p_i y+q_i s+r_i z+w_i\frac{t^\alpha}{\Gamma(1+\alpha)}+c_i, \quad i=1,2. \tag{4.16}$$

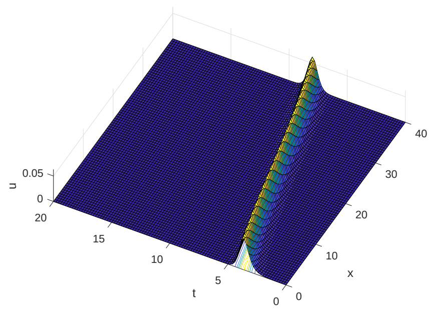
With the help of mathematical software, we can obtain the 3D plots of the single soliton solution (4.11) and double soliton solution (4.15) by selecting the appropriate parameters. Figures 1 and 2 display 3D plots of the single and double soliton solutions in the(x,t)-plane. In 1(a)-(c), we can see the bell-shaped solitary wave under different fractional order α. When α is smaller, the shape of the bell-shaped solitary wave is more affected by the variable coefficient $h_2(t)$, and the shape of the wave is more curved. In addition, as α decreases, the wave's width is wider, and as α increases, the bell-shaped solitary wave moves closer to the $x-$direction. In 2(a)-(c), with the decrease of fractional order α, some similar conclusions can be got for the bell-shaped solitary wave of double soliton as shown in 1. When α is smaller, the shape of the wave is more curved, the wave is gentler, and the wave is more and more deviated from the $x-$direction.



(a) $\alpha=0.5$

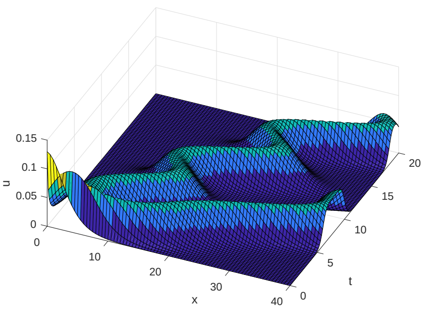


(b) $\alpha=0.75$

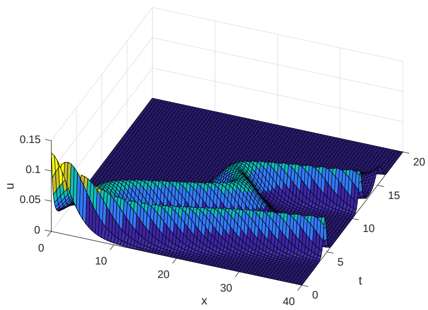


(c) $\alpha=1$

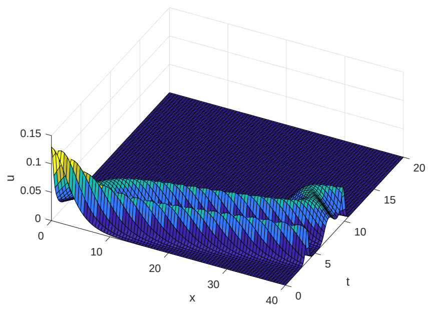
Figure 1: Evolution plots of the solution (4.11) with parameters selected as $k=0.5,p=0.1,q=0.5,r=0.1,c=10,y=s=z=0,g(t)=1,f(t)=12,h_1(t)=h_4(t)=h_5(t)=h_6(t)=h_7(t)=1,h_2(t)=cos(t),h_3(t)=t-3$ for different α: (a) $\alpha=0.5$; (b) $\alpha=0.75$; (c) $\alpha=1$.



(a) $\alpha=0.5$



(b) $\alpha=0.75$



(c) $\alpha=1$

Figure 2: Evolution plots of the solution (4.15) with parameters selected as $k_1=k_2=0.5, p_1=p_2=0.1, q_1=0.5, q_2=1, r_1=0.5, r_2=1, c_1=c_2=y=s=z=0, g(t)=0.1, f(t)=1.2, h_3=h_4=h_5=h_6=h_7=1, h_1=cos(t), h_2=t-5$ for different α: (a) $\alpha=0.5$; (b) $\alpha=0.75$; (c) $\alpha=1$.

5 Numerical results

In this section, by combining the Grünwald-Letnikov method for the time fractional derivative and the Fourier spectral method for the spatial derivative, numerical solutions can be obtained for problems involving both time and space fractional derivatives.

Considering the (4+1)-dimensional KP equation with variable coefficients

$$\begin{aligned}
&D_t^{\alpha}u_x+\frac{f(t)}{2}(u^2)_{xx}+g(t)u_{xxxx}+h_7(t)u_{ss}+h_6(t)u_{zz}+h_5(t)u_{yy}\\
&+h_4(t)u_{xs}+h_3(t)u_{xz}+h_2(t)u_{xy}+h_1(t)u_{xx}=0, \quad (x,y,s,z)\in\Omega\subset R^4, t\in(0,T], \quad (5.1)\\
&u(x,y,s,z,0)=u_0(x,y,s,z), \quad (x,y,s,z)\in\partial\Omega\cup\Omega, t\in(0,T], \quad (5.2)\\
&u(x,y,s,z,t)=\phi(x,y,s,z,t), \quad (x,y,s,z)\in\partial\Omega, t\in(0,T]. \quad (5.3)
\end{aligned}$$

5.1 Time discretization

To discretize the Riemann-Liouville time fractional derivative operator D_t^α using the Grünwald-Letnikov method, we can define the time-step $\tau=\frac{T}{N}$, where N is a positive integer. We also introduce the time points $t_n=n\tau(0\le n\le N)$. Using the Grünwald-Letnikov approximation, the fractional derivative can be expressed as

$$D_t^\alpha p^n \approx \tau^{-\alpha}\sum_{k=0}^{n} w_k^{(\alpha)} p^{n-k}, \tag{5.4}$$

where $w_k^{(\alpha)}=(-1)^k\binom{\alpha}{k}$, and $\binom{\alpha}{k}=\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$. By applying this approximation, we can discretize the fractional derivative operator and obtain the grid function $p=\{p^n|0\le n\le N\}$, which represents the values of the function u at the time points t_n. This discretization allows us to numerically solve the fractional differential equation.

5.2 Space discretization

In space, we suppose the space domain $\Omega=[0,a]\times[0,b]\times[0,c]\times[0,d]$ and spatial mesh size $h_1=\frac{a}{M_1},h_2=\frac{b}{M_2},h_3=\frac{c}{M_3},h_4=\frac{d}{M_4}$, where $h_1=h_2=h_3=h_4$, and M_1,M_2,M_3,M_4 are both integral powers of 2. The gird points can be given as $x_j=jh_1(0\le j\le M_1)$, $y_k=kh_2(0\le k\le M_2)$, $s_l=lh_3(0\le l\le M_3)$, $z_m=mh_4(0\le m\le M_4)$.

Denote the index sets:

$$\begin{aligned}
&\hbar=\{(j,k,l,m)|0<j<M_1,0<k<M_2,0<l<M_3,0<m<M_4\},\\
&\ell=\{j,k,l,m)|0\le j\le M_1,0\le k\le M_2,0\le l\le M_3,0\le m\le M_4\},\\
&\pounds=\{(j,k,l,m)|j=0,or\ j=M_1;or\ k=0,or\ k=M_2;or\ l=0,\\
&\quad or\ l=M_3;or\ m=0,or\ m=M_4\}.
\end{aligned} \tag{5.5}$$

So, each grid point can be represented by its coordinate (j,k,l,m), which corresponds to the specific time point in the discretized time domain. The grid function can be given as $v=\{v_{jklm}|(j,k,l,m)\in\ell\}$.

Denote $S_h=\{v|v=\{v_{jklm}|(j,k,l,m)\in\ell\}$ is the grid function$\}$.

5.3 The numerical scheme

Taking x-direction as an example, in the spectral method, the space derivative in the x-direction can be approximated using the Fourier series expansion. Denoting $k_x=\frac{2\pi r_1}{a}$, where $r_1=-\frac{M_1}{2},-\frac{M_1}{2}+1,\cdots,\frac{M_1}{2}-1$, and there are M_1 grid points in the x- direction when fixed y,z,s and t.

Step 1: Taking out the value of u at each grid node in the x-direction(there are $M_2*M_3*M_4$ columns in totals, one column has M_1 values) and taking the Fast Fourier transform for each column of data. We know that when y,s,z and t are fixed, the $u(x,y_k,s_l,z_m,t_n)$

is a one-dimensional function of x. So the Fast Fourier transform for $u(x,y_k,z_l,s_m,t_n)$ can be given as

$$F_x[u_j] = \sum_{j=0}^{M_1-1} u_j e^{-ik_x x_j}, \tag{5.6}$$

where $k_x = \frac{2\pi r_1}{a}$, $\frac{-M_1}{2} \le r_1 \le \frac{M_1}{2} - 1$.

When performing the Fast Fourier Transform on each column of data, we obtain the Fast Fourier transform of $u(x,y,z,w,t)$ in the x-direction, denoted as $F_x[u]$.

Step 2: The derivative of the Fourier transform for $u(x,y_k,s_l,z_m,t_n)$ is

$$F[(u_{xxxx})_j] = k_x^4 F[u_j], \tag{5.7}$$

so, we have

$$F_x[u_{xxxx}] = k_x^4 F[u]. \tag{5.8}$$

Step 3: Inverse Fast Fourier transform of $u(x,y_k,s_l,z_m,t_n)$ can be given as

$$F^{-1}[u_j] = \frac{1}{M_1} \sum_{r_1=\frac{-M_1}{2}}^{\frac{M_1}{2}-1} F[u_j] e^{ik_x x_j}, 0 \le j \le M_1 - 1, \tag{5.9}$$

where $k_x = \frac{2\pi r_1}{a}$, $\frac{-M_1}{2} \le r_1 \le \frac{M_1}{2} - 1$.

Similarly, by taking the Inverse Fast Fourier Transform of each column of data, we can obtain the Inverse Fast Fourier Transform of $u(x,y,s,z,t)$ in the x-direction, denoted as $F_x^{-1}[u]$. So, for $u_{xxxx}(x,y,s,z,t)$, we have $u_{xxxx} = F_x^{-1}\{k_x^4 F_x[u]\}$.

Similarly, we have

$$\begin{aligned}
&u_{xx} = F_x^{-1}\{-k_x^2 F_x[u]\}, u_{xy} = F_y^{-1}\{-ik_y F_y\{F_x^{-1}\{-ik_x F_x[u]\}\}\},\\
&(u^2)_{xx} = F_x^{-1}\{-k_x^2 F_x[(u)^2]\}, u_{xs} = F_s^{-1}\{-ik_s F_s\{F_x^{-1}\{-ik_x F_x[u]\}\}\},\\
&u_{xz} = F_z^{-1}\{-ik_z F_z\{F_x^{-1}\{-ik_x F_x[u]\}\}\}, u_{yy} = F_y^{-1}\{-k_y^2 F_y[u]\},\\
&u_{ss} = F_s^{-1}\{-k_s^2 F_s[u]\}, u_{zz} = F_z^{-1}\{-k_z^2 F_z[u]\},
\end{aligned} \tag{5.10}$$

where $k_y = \frac{2\pi r_2}{b}, k_s = \frac{2\pi r_3}{c}, k_z = \frac{2\pi r_4}{d}$, $r_2 = -\frac{M_2}{2}, -\frac{M_2}{2}+1, \cdots, \frac{M_2}{2}-1$, $r_3 = -\frac{M_3}{2}, -\frac{M_3}{2}+1, \cdots, \frac{M_3}{2}-1$, $r_4 = -\frac{M_4}{2}, -\frac{M_4}{2}+1, \cdots, \frac{M_4}{2}-1$. $F_y[u]$, $F_s[u]$ and $F_z[u]$ are Fast Fourier transform of $u(x,y,s,z,t)$ in y-direction, s-direction and z-direction respectively.

Consider the Eqs. (5.1)-(5.3) at the point (x_j,y_k,s_l,z_m,t_n). Denoting grid function $\{U_{jklm}^n = u(x_j,y_k,s_l,z_m,t_n) | (j,k,l,m) \in \ell, 0 \le n \le N\}$, and taking $v = D_t^\alpha u$, $V_{jklm}^n = D_t^\alpha U_{jklm}^n$, we have

$$
\begin{aligned}
(V_x)^n_{jklm} = & -\frac{f(t)}{2} F_x^{-1}\{-k_x^2 F_x[(U^n_{jklm})^2]\} - g(t) F_x^{-1}\{k_x^4 F_x[U^n_{jklm}]\} \\
& - h_7(t) F_s^{-1}\{-k_s^2 F_s[U^n_{jklm}]\} - h_6(t) F_z^{-1}\{-k_z^2 F_z[U^n_{jklm}]\} \\
& - h_5(t) F_y^{-1}\{-k_{x_1}^2 F_y[U^n_{jklm}]\} - h_4(t) F_s^{-1}\{-ik_s F_s\{F_x^{-1}\{-ik_x F_x[U^n_{jklm}]\}\}\} \\
& - h_3(t) F_z^{-1}\{-ik_z F_z\{F_x^{-1}\{-ik_x F_x[U^n_{jklm}]\}\}\} \\
& - h_2(t) F_y^{-1}\{-ik_y F_y\{F_x^{-1}\{-ik_x F_x[U^n_{jklm}]\}\}\} \\
& - h_1(t) F_x^{-1}\{-k_x^2 F_x[U^n_{jklm}]\}, \qquad (j,k,l,m) \in \hbar, 1 \le n \le N,
\end{aligned}
\tag{5.11}
$$

$$
U^0_{jklm} = u_0(x_j, y_k, s_l, z_m), \qquad (j,k,l,m) \in \ell, \tag{5.12}
$$

$$
U^n_{jklm} = \phi(x_j, y_k, s_l, z_m, t_n), \qquad (j,k,l,m) \in \pounds, 0 \le n \le N. \tag{5.13}
$$

For the sake of simplicity, we have

$$
\begin{aligned}
A^n = & -\frac{f(t)}{2} F_x^{-1}\{-k_x^2 F_x[(U^n_{jklm})^2]\} - g(t) F_x^{-1}\{k_x^4 F_x[U^n_{jklm}]\} \\
& - h_7(t) F_s^{-1}\{-k_s^2 F_s[U^n_{jklm}]\} - h_6(t) F_z^{-1}\{-k_z^2 F_z[U^n_{jklm}]\} \\
& - h_5(t) F_y^{-1}\{-k_{x_1}^2 F_y[U^n_{jklm}]\} \\
& - h_4(t) F_s^{-1}\{-ik_s F_s\{F_x^{-1}\{-ik_x F_x[U^n_{jklm}]\}\}\} \\
& - h_3(t) F_z^{-1}\{-ik_z F_z\{F_x^{-1}\{-ik_x F_x[U^n_{jklm}]\}\}\} \\
& - h_2(t) F_y^{-1}\{-ik_y F_y\{F_x^{-1}\{-ik_x F_x[U^n_{jklm}]\}\}\} \\
& - h_1(t) F_x^{-1}\{-k_x^2 F_x[U^n_{jklm}]\}, \qquad (j,k,l,m) \in \hbar, 1 \le n \le N,
\end{aligned}
\tag{5.14}
$$

So

$$
(V_x)^n_{jklm} = A^n, \qquad (j,k,l,m) \in \hbar, 1 \le n \le N, \tag{5.15}
$$

$$
U^0_{jklm} = u_0(x_j, y_k, s_l, z_m), \qquad (j,k,l,m) \in \ell, \tag{5.16}
$$

$$
U^n_{jklm} = \phi(x_j, y_k, s_l, z_m, t_n), \qquad (j,k,l,m) \in \pounds, 0 \le n \le N. \tag{5.17}
$$

By applying the Fast Fourier Transform and the Inverse Fourier Transform to both sides of Eq. (5.1) with respect to the x-direction, we obtain

$$
-ik_x F_x[V^n_{jklm}] = F_x[A^n], \qquad (j,k,l,m) \in \hbar, 1 \le n \le N, \tag{5.18}
$$

$$
V^n_{jklm} = F_x^{-1}\left\{\frac{F_x[A^n]}{-ik_x}\right\}, \qquad (j,k,l,m) \in \hbar, 1 \le n \le N, \tag{5.19}
$$

$$
U^n_{jklm} \approx \left(\tau^\alpha V^n_{jklm} - \sum_{k=1}^{n} w_k^{(\alpha)} U^{n-k}_{ijlm}\right) \Big/ w_0^{(\alpha)}, \qquad (j,k,l,m) \in \hbar, 1 \le n \le N, \tag{5.20}
$$

$$
U^0_{jklm} = u_0(x_j, y_k, s_l, z_m), \qquad (j,k,l,m) \in \ell, \tag{5.21}
$$

$$U_{jklm}^{n}=\phi(x_j,y_k,s_l,z_m,t_n), i=0, \qquad (j,k,l,m)\in \pounds, 0\leq n\leq N. \tag{5.22}$$

Replacing U_{jklm}^{n} with u_{jklm}^{n} and replacing V_{jklm}^{n} with v_{jklm}^{n}, so we have

$$-ik_xF_x[v_{jklm}^{n}]=F_x[A^n], \qquad (j,k,l,m)\in \hbar, 1\leq n\leq N, \tag{5.23}$$

$$v_{jklm}^{n}=F_x^{-1}\left\{\frac{F_x[A^n]}{-ik_x}\right\}, \qquad (j,k,l,m)\in \hbar, 1\leq n\leq N, \tag{5.24}$$

$$u_{jklm}^{n}=\left(\tau^{\alpha}v_{jklm}^{n}-\sum_{k=1}^{n}w_k^{(\alpha)}u_{jklm}^{n-k}\right)\Big/w_0^{(\alpha)}, \qquad (j,k,l,m)\in \hbar, 1\leq n\leq N, \tag{5.25}$$

$$u_{jklm}^{0}=u_0(x_j,y_k,s_l,z_m), \qquad (j,k,l,m)\in \ell, \tag{5.26}$$

$$u_{jklm}^{n}=\phi(x_j,y_k,s_l,z_m,t_n), \qquad (j,k,l,m)\in \pounds, 0\leq n\leq N. \tag{5.27}$$

5.4 Numerical results

We provide two examples to demonstrate the effectiveness of our proposed numerical method discussed in subsection 5.3.

Example 5.1. When we consider each variable coefficient of Eq. (2.10) as 1, we can obtain the equation

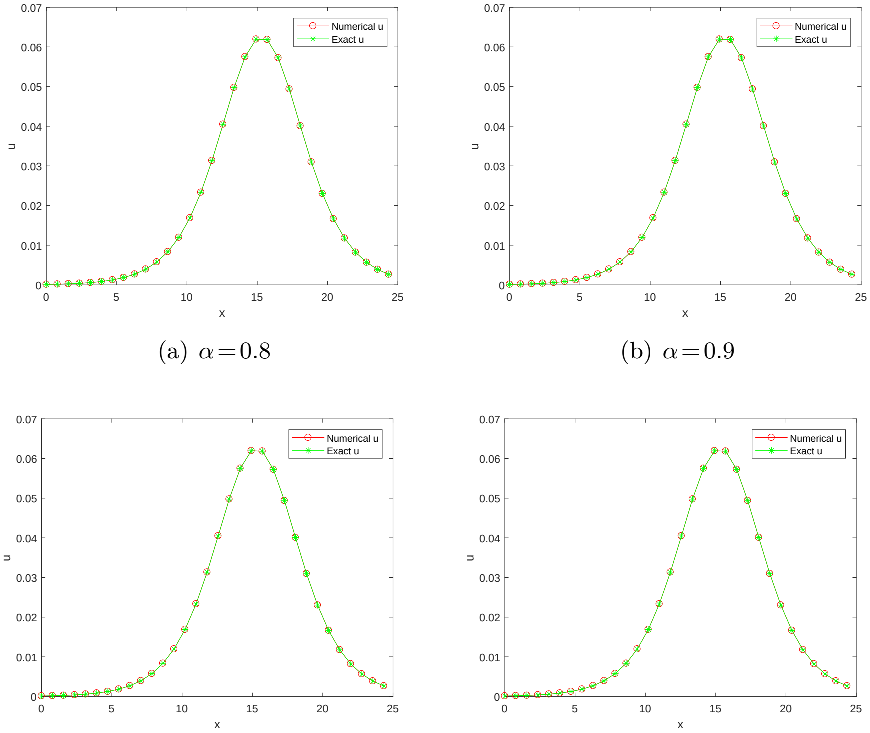
$$D_t^{\alpha}u_x+(u^2)_{xx}+u_{xxxx}+u_{ss}+u_{zz}+u_{yy}+u_{xs}+u_{xz}+u_{xy}+u_{xx}=0,$$

where $(x,y,s,z)\in R^4, t\in(0,T]$. By appropriately selecting free parameters from Eq. (4.11), we can obtain the exact solution for the above equation.

The initial conditions and boundary conditions are determined by Eq. (4.11). We compare the exact solution given by Hirota bilinear method with the numerical solution given by pseudo-spectral method to demonstrate the effectiveness of the proposed numerical method. When taking $\alpha=0.8$, $\alpha=0.9$, $\alpha=0.98$ and $\alpha=1$, the maximum absolute errors of exact solutions and numerical solutions under different fractional orders are given in Table 1, and we give two-dimensional comparison images of exact and numerical solutions under different fractional orders in Figure 3(a)-(d). The results of error and curve fitting are acceptable, which also show the accuracy of the proposed pseudo-spectral method.

Table 1: The maximum absolute errors between the numerical solutions and the exact solutions in Eq. (4.11) for different fractional order α.

α	**Errors**	α	**Errors**
0.8	2.8065E-07	0.98	2.7936E-08
0.9	7.1646E-08	1	1.4624E-11



(a) $\alpha=0.8$ (b) $\alpha=0.9$

(c) $\alpha=0.98$ (d) $\alpha=1$

Figure 3: Comparison of the numerical solutions and the exact solutions in Eq. (4.11) at the end time for different fractional order α with $y=\pi, s=2\pi, z=3\pi$. (a) $\alpha=0.8$; (b) $\alpha=0.9$; (c) $\alpha=0.98$; (d) $\alpha=1$.

Example 5.2. When we consider each variable coefficient of Eq. (2.10), $h_1(t)=h_5(t)=h_6(t)=h_7(t)=-1$, $f(t)=g(t)=h_4(t)=h_3(t)=h_2(t)=0$, we can obtain the equation

$$D_t^\alpha u_x = u_{xx}+u_{yy}+u_{ss}+u_{zz},$$

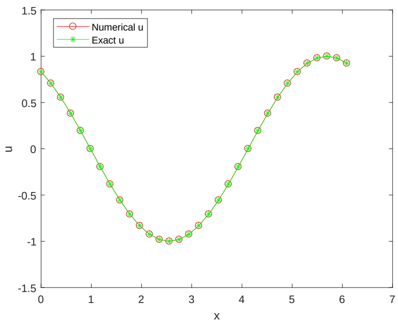
where $(x,y,s,z)\in[0,2\pi]\times[0,2\pi]\times[0,2\pi]\times[0,2\pi], t\in(0,T]$, and the exact solution of above equation is

$$u(x,y,s,z,t)=\sin\left(x+y+z+s+\frac{4}{\Gamma(1+\alpha)}t^\alpha\right),$$

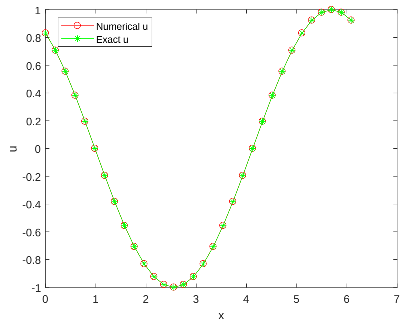
in which, $\Gamma(x)$ is the standard Gamma function.

Both initial and boundary conditions are derived from the exact solution. Table 2 shows the maximum absolute error between numerical and exact solutions under different fractional order α. Figure 4 illustrates the comparison between the numerical and exact solutions for different fractional orders. The results presented in Table 2 and Figure 4 demonstrate that the error values and curve fitting results for different fractional orders

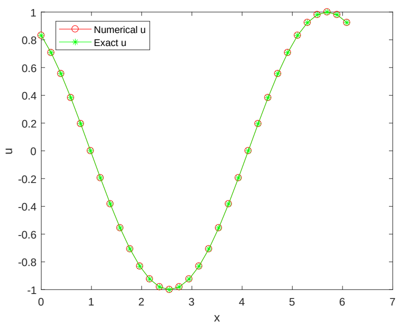
α are acceptable and satisfactory. These findings indicate the feasibility of our proposed numerical method.



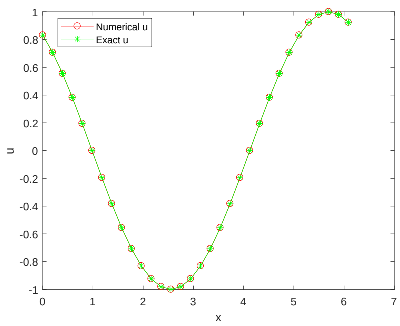
(a) $\alpha = 0.8$



(b) $\alpha = 0.9$



(c) $\alpha = 0.98$



(d) $\alpha = 1$

Figure 4: Comparison of the numerical solutions and the exact solution at the end time for different fractional order α with $y = \pi/8, s = \pi/4, z = \pi/2$. (a) $\alpha = 0.8$; (b) $\alpha = 0.9$; (c) $\alpha = 0.98$; (d) $\alpha = 1$.

Table 2: The maximum absolute errors between the numerical solutions and the exact solutions for different fractional order α.

α	**Errors**	α	**Errors**
0.8	1.6965E-05	0.98	1.0666E-06
0.9	3.8329E-06	1	8.0000E-09

6 Conclusions

In this study, we have investigated the (4+1)-dimensional time fractional KP equation with variable coefficients. The equation is considered in the sense of Riemann-Liouville fractional derivative, which allows us to model systems with fractional order dynamics.To analyze the equation further, we employed the Lie symmetry analysis method. This mathematical technique helps identify the symmetries of the equation, which are crucial for understanding its behavior and properties. By applying Lie symmetry and the adjoint equation, we were able to derive the conservation laws of the equation with variable coefficients. Then we explored the solutions of the equation using different methods. First, we utilized the Hirota method to obtain soliton solutions. Solitons are localized and stable waveforms that propagate without changing their shape. Additionally, we employed the Pseudo-spectral method to obtain numerical solutions. This method is commonly used for solving partial differential equations numerically, providing accurate results by utilizing high-order approximations. To assess the effectiveness of the numerical method, we calculated error results and compared images of the solutions. These evaluations demonstrated that the numerical method is capable of accurately capturing the dynamics of the equation with variable coefficients. This paper presents a comprehensive analysis of the (4+1)-dimensional time fractional KP equation with variable coefficients, including symmetry analysis, conservation laws, and various solution methods. The results provide valuable insights into the behavior of this equation and pave the way for further research in fractional differential equations.

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Conflicts of Interest

The authors declare no conflict of interest.

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