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Numerical Assessment of a Class of High Order Stokes Spectrum Solver

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Abstract. It is well known that the approximation of eigenvalues and associated eigenfunctions of a linear operator under constraint is a difficult problem. One of the difficulties is to propose methods of approximation which satisfy in a stable and accurate way the eigenvalues equations, the constraint one and the boundary conditions. Using any non-stable method leads to the presence of non-physical eigenvalues: a multiple zero one called spurious modes and non-zero one called pollution modes. One way to eliminate these two families is to favor the constraint equations by satisfying it exactly and to verify the equations of the eigenvalues equations in weak ways. To illustrate our contribution in this field we consider in this paper the case of Stokes operator. We describe several methods that produce the correct number of eigenvalues. We numerically prove how these methods are adequate to correctly solve the 2D Stokes eigenvalue problem.

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1 Introduction

The 2D Stokes eigenvalue problem on a square domain is considered in this paper as model example with a conservation law of the type $\nabla \cdot \boldsymbol{u} = 0$. With this test example it is possible to discuss the various numerical problems that appear when flux conservation has to be satisfied in the incompressible Navier-Stokes. If these constraint condition cannot be satisfied precisely, so-called spectral pollution [9] appears and the numerical

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approach does not stably converge to the physical solution. The reason is that due to regularity constraints imposed by standard numerical approximation methods, the energy cannot reach the minimum required by the physics. In fact, current numerical methods satisfy the boundary conditions strongly, the operator equations and the constraints only weakly.

In Section 2, we propose a non-exhaustive list of methods to deal with the 2D Stokes eigenvalue problem. Specifically, if the constraint $\nabla\cdot\boldsymbol{u}=0$ is satisfied by a $\boldsymbol{u}=\nabla\times\psi$ ansatz, the number of degrees of freedom remains the same as in the unconstrained Laplacian problem. As a consequence, besides the Stokes modes, one finds a whole spectrum of additional unphysical modes, corresponding to those of the heat equation. Thus, the initial physical problem has fundamentally been changed. This approach has been applied to compute the full Stokes spectrum [1] by the first time. Due to the choice of a unit square domain, the authors were able to separate the Stokes modes from those belonging to the heat equation. An other strategy consists in applying a penalty method to solve the Stokes problem. In this case, the number of degrees of freedom still remains the same as those in the unconstrained Laplacian problem.

In Section 3, we focus on the two formulations considering only the velocity as variable: the penalty method and the divergence-free Galerkin approach. In the framework of spectral element approximation schemes, a stable spectral element is proposed for each method. For the penalty method, a COOL approach [2,3] is made and the unphysical modes can be pushed towards $\lambda=0$. For the divergence-free Galerkin approach, two strategies christened "explicit" and "implicit" are detailed. The explicit strategy consists in using the properties of the kernel of the **grad**(div) operator to construct a divergence-free basis. Such a basis has the right number of degrees of freedom, thus delivering the exact number of Stokes eigenfunctions with high precision. The implicit strategy is a direct algebraic elimination process of the $\nabla\cdot\boldsymbol{u}=0$ constraint. This leads to a sparse matrix elimination process, described in detail in [2]. It delivers the right number of highly precise Stokes modes.

Finally, in Section 4, some numerical experiments are performed to prove the efficiency of the proposed methods and a comparison between the different approaches is given.

2 The Stokes eigenvalue problem: continuous version

Let $\Omega\subset IR^d$, $d=2,3$, be a Lipschitz domain, the generic point of Ω is denoted $\boldsymbol{x}$. The symbol $L^2(\Omega)$ stands for the usual Lebesgue space and $H^1(\Omega)$, the Sobolev space that involves all the functions that are, together with their gradient, in $L^2(\Omega)$. $\mathcal{C}(\Omega)$ denotes the space of continuous functions defined in Ω.

The continuous Stokes eigenvalue problem reads: *Find a vector* $\boldsymbol{u}$ *and* $\lambda^2\in IR^+$ *such*