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Some Properties of Solutions to the Novikov Equation with Weak Dissipation Terms

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Abstract In this paper, we investigate the Novikov equation with weak dissipation terms. First, we give the local well-posedness and the blow-up scenario. Then, we discuss the global existence of the solutions under certain conditions. After that, on condition that the compactly supported initial data keeps its sign, we prove the infinite propagation speed of our solutions, and establish the large time behavior. Finally, we also elaborate the persistence property of our solutions in weighted Sobolev space.

Keywords Blow-up scenario, Global existence, Large time behavior, Persistence property.

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1. Introduction

In this paper, we discuss the following Novikov equation with weak dissipation terms:

$$y_t + y_x u^2 + byuu_x + \lambda y = 0, \quad t > 0, x \in \mathbb{R}. \tag{1.1}$$

When $\lambda = 0$, it is a special case of the Holm-Staley b-family equations:

$$y_t + y_x u^k + byu^{k-1}u_x = 0, \quad t > 0, x \in \mathbb{R}, \tag{1.2}$$

where $k \geqslant 1$, $\lambda \in \mathbb{R}$, $u(x,t)$ denote the velocity field, $y(x,t) = u - u_{xx}$.

Holm and Staley [28] got the exchange of stability in the dynamics of solitary wave solutions under changes in the nonlinear balance, which was in a 1+1 evolutionary partial differential equation both related to shallow water waves and to turbulence.

When $k = 1$ and $b = 2$, equation (1.2) reduces to the famous Camassa-Holm equation [4], while, if $k = 1$ and $b = 3$, it reduces to the Degasperis-Procesi equation. These equations arise at various levels of approximation in shallow water theory, and possess a physics background with shallow water propagation, the bi-Hamiltonian structure, Lax pair and explicit solutions including classical soliton, cuspon and

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peakon solutions. Moreover, these two types of equations have been also extensively studied in [5, 10, 11, 13, 18, 19, 22, 26, 27, 39, 42, 53].

We know that Camassa-Holm equation is completely integrable. Definitely, the Camassa-Holm equation has many useful properties, for example, conservation rate, blow-up scenario Global existence and large time behavior for the support of the momentum density [30,40]. When it comes to the physical relevance of the Camassa-Holm and Degasperis-Procesi equation, we suggest the readers reading the book written by Constantin and Lannes [14]. In the $H^s, s > \frac{3}{2}$ space, the solution of the local well-posedness was proved in [11,36]. In [11,12,32,36,41], the blow up scenario was widely used. For the Camassa-Holm, the solution of Global existence and local solution was proved in [2, 3, 33]. They also proved orbital stability of the peak solution in [15], In [27], Himonas et al., gave the persistence and unique continuity of solution of Cassama-Holm equation. They discussed the large time behavior for the support of momentum density of the Camassa-Holm equation. They proved the limit of the support of momentum density as t goes to $+\infty$ in some sense. Moreover, the Degasperis-Procesi equation has been widely studied in [8, 9, 17, 31, 37, 44, 53].

When $k = 1$, for general b, the equation (1.2) was studied in [21, 54], which has established the local well-posedness and sufficient conditions on the initial data to guarantee the global existence of strong solutions in H^s, $s > \frac{3}{2}$. Blow-up scenario for equation (1.2) has been studied in [20, 43, 45, 47, 52], and some blow-up criteria was established in [16, 50]. Guan and Yin [24] studied the global existence and blow-up phenomenon of the integrable two-component Camassa-Holm shallow water system. Moreover, Liu and Yin [55] presented several conditions for the existence of global solutions. The large-time behavior of the supporting the momentum density for the Camassa-Holm equation was studied in [33]. [49] proposed a new method to show the persistence properties. Guo et al., [25] studied the large time behavior and persistence properties of solutions to the Camassa-Holm-type equation with higher-order nonlinearities. Here, we would like mention some related work of equation (1.2) in [7, 23, 29, 35, 38, 48, 51, 57].

In 2011, Zhu and Jiang [59] discussed the case of $k = 1$ in (1.2):

$$y_t + y_x u + byu_x + \lambda y = 0, \quad t > 0, x \in \mathbb{R}, \tag{1.3}$$

and got a new criterion on the blow-up phenomenon of the solution, the global existence and the persistence property of the solution. Zhang [56] considered the Camassa-Holm equation with weak dissipation terms. Niu and Zhang [45] established the local well-posedness of the inhomogeneous weak dissipation equation, which included both the weakly dissipative Camassa-Holm equation and the weakly dissipative Degasperis-Procesi equation as its special case. Zhou et al., [58] discussed the following more general equation:

$$y_t + y_x u^k + byu^{k-1}u_x + \lambda y = 0, \quad t > 0, x \in \mathbb{R}. \tag{1.4}$$

For equation (1.4), Zhou et al., [6, 34, 58] listed some existing conditions of global solution and some analytical properties of solution. When $k = 2$, the equation (1.4) would be the equation (1.1). The equation (1.1) could be rewritten as:

$$u_t + u^2 u_x + G * F(u) + \lambda u = 0, \tag{1.5}$$

where

$$F(u) = (6 - b)uu_x u_{xx} + 2u_x^3 + bu^2 u_x, \tag{1.6}$$

where $y = (1 - \partial_x^2)u$ is usually called the potential of fluid.

We organize this paper as follows: First, we give the local well-posedness and the blow-up scenario of the solution in Section 2. Next, in Section 3, we discuss the global existence under certain conditions. Then, we prove the infinite propagation speed and establish the large time behavior properties of our solution in Section 4 and Section 5. Finally, we elaborate the persistence property in Section 6.

2. Local well-posedness and blow-up scenario

In this section, we give the local well-posedness of the equation (1.1) first. Then, we show the blow-up scenario for the solution to (1.1).

Lemma 2.1. *Give the $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then, there exist a $T > 0$ and a unique solution $u(x,t)$ to* (1.1) *such that*

$$u(x,t) \in C([0,T); H^s(R)) \cap C^1([0,T); H^{1/2}(\mathbb{R})). \tag{2.1}$$

Moreover, the map $u_0(x) \in H^s \to u \in C([0,T); H^s(\mathbb{R}))$ is continuous but not uniformly continuous.

To prove this result, we will apply Kato's theorem [46], with $X = H^{1/2}$, $Y = H^s$, $S = \Lambda^{s-1/2}$, $A(u) = u^2\partial_x$, $f(u) = (b-6)uu_xu_{xx} - 2u_x^3 - bu^2u_x$ and $W = \{\varphi \in H^s | \|\varphi\|_{H^s} \leqslant R\}$.

Moreover, we obtain that $u(x,t) \in C([0,T); H^s(R)) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$, because $u_t \in H^{s-1}$.

Theorem 2.1. *Assume that $u_0 \in H^2(\mathbb{R})$, and let T be the maximal existence time of the solution $u(x,t)$ to the equation* (1.1) *with the initial data $u_0(x)$.*

(1) If $b > 1$, then the corresponding solution of the equation (1.1) *blows up in finite time, if and only if*

$$\lim_{t\to T}\int_0^t \sup_{x\in\mathbb{R}}(uu_x)ds = -\infty. \tag{2.2}$$

(2) If $b < 1$, then the corresponding solution of the equation (1.1) *blows up in finite time, if and only if*

$$\lim_{t\to T}\int_0^t \inf_{x\in\mathbb{R}}(uu_x)ds = +\infty. \tag{2.3}$$

Proof. (1) If $b > 1$, by applying y on (1.1), we have

$$yy_t + yy_xu^2 + by^2uu_x + \lambda y^2 = 0$$

Integrating it with respect to x in $\mathbb{R}$, we have

$$\int_{\mathbb{R}} yy_t dx = -\int_{\mathbb{R}} yy_xu^2ydx - b\int_{\mathbb{R}} y^2uu_xdx - \lambda\int_{\mathbb{R}} y^2dx,$$

which is

$$\frac{1}{2}\frac{d}{dt}\|y\|_{L^2}^2 = \int_{\mathbb{R}} \frac{1}{2}(y^2)_t dx$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}} \frac{1}{2} u^2 (y^2)_x dx - b \int_{\mathbb{R}} y^2 u u_x dx - \lambda \int_{\mathbb{R}} y^2 dx \\
&= -\frac{1}{2} u^2 y^2 \Big|_{\mathbb{R}} + \int_{\mathbb{R}} y^2 u u_x dx - b \int_{\mathbb{R}} y^2 u u_x dx - \lambda \int_{\mathbb{R}} y^2 dx \\
&= (1-b) \int_{\mathbb{R}} y^2 u u_x dx - \lambda \|y\|_{L^2}^2 ,
\end{aligned}
$$

implying

$$\frac{d}{dt} \|y\|_{L^2}^2 + 2\lambda \|y\|_{L^2}^2 = 2(1-b) \int_{\mathbb{R}} y^2 u u_x dx.$$

Letting $M(t) = \sup_{x \in \mathbb{R}} (u u_x)$, we have

$$\frac{d}{dt} \|y\|_{L^2}^2 \geqslant [2(1-b)M(t) - 2\lambda] \|y\|_{L^2}^2 .$$

Then, we obtain

$$\|y\|_{L^2}^2 \geqslant \|y_0\|_{L^2}^2 e^{2(1-b) \int_0^t (M(s) - \frac{\lambda}{1-b}) ds}.$$

For any finite time T, when $t \in [0, T]$, if and only if

$$\lim_{t \to T} \int_0^t \sup_{x \in \mathbb{R}} (u u_x) ds = -\infty.$$

Then, the corresponding solution of the equation (1.1) blows up in finite time.

(2) If $b < 1$, let $m(t) = \inf_{x \in \mathbb{R}} (u u_x)$, by using similar method that is used in case (1), we have

$$\frac{d}{dt} \|y\|_{L^2}^2 \geqslant [2(1-b)m(t) - 2\lambda] \|y\|_{L^2}^2 .$$

It follows that

$$\|y\|_{L^2}^2 \geqslant \|y_0\|_{L^2}^2 e^{2(1-b) \int_0^t (m(s) - \frac{\lambda}{1-b}) ds}.$$

For any finite time T, when $t \in [0, T]$, if and only if

$$\lim_{t \to T} \int_0^t \inf_{x \in \mathbb{R}} (u u_x) ds = +\infty.$$

Then, the corresponding solution of the equation (1.1) blows up in finite time.

□

3. Global existence

In this section, we discuss some global existence. Now, we introduce the particle trajectory. Let $u(x, t)$ be a strong solution of (1.1) obtained in the local well-posedness theorem.

Assume that $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$, by using the Green function $G = \frac{1}{2}e^{-|x|}$, we have

$$u(x,t) = \Lambda^{-2}y(x,t) = G * y(x,t) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\xi|}y(\xi,t)d\xi,$$

which is

$$\begin{cases} u(x,t) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi, \\ u_x(x,t) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi. \end{cases} \tag{3.1}$$

It follows

$$\begin{cases} u(x,t) + u_x(x,t) = e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi, \\ u(x,t) - u_x(x,t) = e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi. \end{cases} \tag{3.2}$$

Let $q(x,t)$ be the solution of the following initial value problem

$$\begin{cases} \frac{dq(x,t)}{dt} = u^2(q(x,t),t), \quad t\in[0,T], \quad x\in\mathbb{R}. \\ q(x,0) = x, \quad x\in\mathbb{R}. \end{cases} \tag{3.3}$$

Taking derivative with respect to x, we have

$$\frac{dq_t(x,t)}{dx} = 2uu_x(q,t)q_x.$$

Then, it follows that

$$\begin{cases} q_x(x,t) = \exp\{\int_0^t 2uu_x(q,s)ds\}, \quad t\in[0,T], \quad x\in\mathbb{R}. \\ q_x(x,0) = 1, \quad x\in\mathbb{R}. \end{cases}$$

Therefore, $q(x,t) : \mathbb{R}\to\mathbb{R}$ is an increasing diffeomorphism of the line before blow-up. From (1.1), by direct calculation, we have

$$\begin{aligned} \frac{d}{dt}(y(q(x,t),t)q_x^{\frac{b}{2}}) &= [y_t(q(x,t),t) + u^2(q(x,t),t)y_x(q(x,t),t) \\ &\quad + buy(q(x,t),t)u_x(q(x,t),t)]q_x^{\frac{b}{2}} \\ &= -\lambda y(q(x,t),t)q_x^{\frac{b}{2}}. \end{aligned}$$

Then, we can prove the following pointwise conservation law

$$y(q(x,t),t)q_x^{\frac{b}{2}}(x,t) = y_0(x)e^{-\lambda t}. \tag{3.4}$$

From (3.4), we can easily obtain

$$e^{\frac{2\lambda t}{b}}\int_{\mathbb{R}} y^{\frac{2}{b}}dx = \int_{\mathbb{R}} y_0^{\frac{2}{b}}dx. \tag{3.5}$$

Theorem 3.1. *Assume that* $u_0 \in H^2(\mathbb{R})$, *if* $b = 1$ *and* $\lambda \geqslant 0$, *the solution of equation* (1.1) *exists globally in time.*

Proof. Applying y on (1.1) and taking integral with respect to x, we have

$$\frac{d}{dt}\|y\|_{L^2}^2 + 2\lambda\|y\|_{L^2}^2 = 2(1-b)\int_{\mathbb{R}} y^2 u u_x dx.$$

If $b = 1$, we obtain

$$\frac{d}{dt}(e^{2\lambda t}\|y\|_{L^2}^2) = e^{2\lambda t}(\frac{d}{dt}\|y\|_{L^2}^2 + 2\lambda\|y\|_{L^2}^2) = 0.$$

When $\lambda \geqslant 0$, we have

$$\|y\|_{L^2}^2 = e^{-2\lambda t}\|y_0\|_{L^2}^2 \leqslant \|y_0\|_{L^2}^2.$$

It follows that

$$\|u\|_{L^2}^2 \leqslant \|y\|_{L^2}^2 \leqslant \|y_0\|_{L^2}^2 \leqslant C\|u_0\|_{L^2}^2.$$

Then, by using theorem 2.1, we prove the global existence of the solution for the equation (1.1). □

Theorem 3.2. *Suppose that* $u_0 \in H^2(\mathbb{R}), y_0 = (1-\partial_x^2)u_0$ *does not change sign, then we have*

(1) if $b = 2$ *and* $\lambda \geqslant 0$, *the corresponding solution to* (1.1) *exists globally.*

(2) if $b < 2$ *and* $\lambda \geqslant 0$, *the corresponding solution to* (1.1) *exists globally.*

Proof. (1) Taking integral both sides of the equation (1.1) to x variable, we have

$$\int_{\mathbb{R}} y_t dx = -\int_{\mathbb{R}} y_x u^2 dx - b\int_{\mathbb{R}} y u u_x dx - \lambda\int_{\mathbb{R}} y dx.$$

It follows that

$$\frac{d}{dt}\|y\|_{L^1} + \lambda\|y\|_{L^1} = (2-b)\int_{\mathbb{R}} y u u_x dx.$$

If $b = 2$, by applying $e^{\lambda t}$ on the formula above, we obtain

$$\frac{d}{dt}(e^{\lambda t}\|y\|_{L^1}) = e^{\lambda t}(\frac{d}{dt}\|y\|_{L^1} + \lambda\|y\|_{L^1}) = 0.$$

When $\lambda \geqslant 0$, we get

$$\|y\|_{L^1} = e^{-\lambda t}\|y_0\|_{L^1} \leqslant \|y_0\|_{L^1}.$$

Assume that $y_0 \geqslant 0$, by the blow-up scenario, it is sufficient that u and u_x are bounded for all t. By equation (3.1), we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} y(\xi,t)d\xi + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi} y(\xi,t)d\xi \\ &\leqslant \int_{\mathbb{R}} y(\xi,t)d\xi \end{aligned}$$

$$
\begin{aligned}
&= e^{-\lambda t}\int_{\mathbb{R}} y_0(\xi,t)d\xi \\
&\leqslant \int_{\mathbb{R}} y_0(\xi,t)d\xi
\end{aligned}
$$

and

$$
\begin{cases}
u_x \leqslant \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi \leqslant \frac{1}{2}\int_{\mathbb{R}} y(\xi,t)d\xi \leqslant \frac{1}{2}\int_{\mathbb{R}} y_0(\xi,t)d\xi, \\
u_x \geqslant -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi \geqslant -\frac{1}{2}\int_{\mathbb{R}} y(\xi,t)d\xi \geqslant -\frac{1}{2}\int_{\mathbb{R}} y_0(\xi,t)d\xi.
\end{cases}
$$

Based on the calculation above, we have

$$
\begin{cases}
0 \leqslant u \leqslant \int_{\mathbb{R}} y_0(\xi,t)d\xi, \\
-\frac{1}{2}\int_{\mathbb{R}} y_0(\xi,t)d\xi \leqslant u_x \leqslant \frac{1}{2}\int_{\mathbb{R}} y_0(\xi,t)d\xi.
\end{cases}
$$

Hence, the conclusion is established, when $y_0 \leqslant 0$. By the similar method that is applied above, we could also obtain the global existence result. Therefore, when $y_0 = (1-\partial_x^2)u_0$ does not change sign, the solution of equation (1.1) exists globally.

(2) If $b < 2$ and $\lambda \geqslant 0$, by (3.5), assume that $y_0 \geqslant 0$, we have

$$
\begin{aligned}
& e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi \\
&\leqslant e^{-x}\left(\int_{-\infty}^{x} y^{\frac{2}{b}}(\xi,t)d\xi\right)^{\frac{b}{2}}\left(\int_{-\infty}^{x} e^{\frac{2\xi}{2-b}}d\xi\right)^{\frac{2-b}{2}} \\
&= e^{-\lambda t}\left(\frac{2-b}{2}\right)^{\frac{2-b}{2}}\left(\int_{-\infty}^{x} y_0^{\frac{2}{b}}(\xi,t)d\xi\right)^{\frac{b}{2}} \\
&\leqslant \left(\frac{2-b}{2}\right)^{\frac{2-b}{2}}\left(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi\right)^{\frac{b}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
& e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi \\
&\leqslant e^{x}\left(\int_{x}^{+\infty} y^{\frac{2}{b}}(\xi,t)d\xi\right)^{\frac{b}{2}}\left(\int_{x}^{+\infty} e^{\frac{-2\xi}{2-b}}d\xi\right)^{\frac{2-b}{2}} \\
&= e^{-\lambda t}\left(\frac{2-b}{2}\right)^{\frac{2-b}{2}}\left(\int_{x}^{+\infty} y_0^{\frac{2}{b}}(\xi,t)d\xi\right)^{\frac{b}{2}} \\
&\leqslant \left(\frac{2-b}{2}\right)^{\frac{2-b}{2}}\left(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi\right)^{\frac{b}{2}}.
\end{aligned}
$$

By the blow-up scenario, it is sufficient that u and u_x are bounded for all t. Therefore, by equation (3.1), we have

$$
u(x,t) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi
$$

$$\leqslant (\frac{2-b}{2})^{\frac{2-b}{2}}(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi)^{\frac{b}{2}}$$

and

$$\begin{aligned} u_x(x,t) &\leqslant \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi,t)d\xi \\ &\leqslant \frac{1}{2}(\frac{2-b}{2})^{\frac{2-b}{2}}(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi)^{\frac{b}{2}}. \end{aligned}$$

$$\begin{aligned} u_x(x,t) &\geqslant -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi \\ &\geqslant -\frac{1}{2}(\frac{2-b}{2})^{\frac{2-b}{2}}(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi)^{\frac{b}{2}}. \end{aligned}$$

Based on the calculation above, we have

$$\begin{cases} 0 \leqslant u \leqslant (\frac{2-b}{2})^{\frac{2-b}{2}}(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi)^{\frac{b}{2}}, \\ -\frac{1}{2}(\frac{2-b}{2})^{\frac{2-b}{2}}(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi)^{\frac{b}{2}} \leqslant u_x \leqslant \frac{1}{2}(\frac{2-b}{2})^{\frac{2-b}{2}}(\int_{\mathbb{R}} y_0^{\frac{2}{b}}(\xi,t)d\xi)^{\frac{b}{2}}. \end{cases}$$

Hence, the conclusion is established, when $y_0 \leqslant 0$. By the similar method that is used above, we also get the global existence result. Therefore, when $y_0 = (1-\partial_x^2)u_0$ does not change sign, the soltion of equation (1.1) exists globally. □

4. Infinite propagation speed

Set

$$E(t) = \int_{\mathbb{R}} e^{\xi}y(\xi,t)d\xi, \qquad F(t) = \int_{\mathbb{R}} e^{-\xi}y(\xi,t)d\xi. \tag{4.1}$$

Theorem 4.1. *If $b \geqslant 0, \lambda \in \mathbb{R}$, assume the initial value $u_0 \not\equiv 0$ has a compact supported set $[a,c]$. For $t \in (0,T]$, the solution $u(x,t)$ corresponding to the* (1.1) *has the following properties:*

$$u(x,t) = \begin{cases} \frac{1}{2}e^{-x}E(t) & ,if \quad x > q(c,t). \\ \frac{1}{2}e^{x}F(t) & ,if \quad x < q(a,t). \end{cases} \tag{4.2}$$

In addition, if $b \in [0,6]$, we have

(1) when $y_0 \geqslant 0$, $e^{\lambda t}E(t)$ is a strictly increasing function, $e^{\lambda t}F(t)$ is a strictly decreasing function.

(2) when $y_0 \leqslant 0$, $e^{\lambda t}E(t)$ is a strictly decreasing function, $e^{\lambda t}F(t)$ is a strictly increasing function.

Remark 4.1. From the theorem above we know, even if the initial value u_0 has a compact supported set $[a,c]$, for any $t > 0$, the solution $u(x,t)$ does not have a compact supported.

Proof. From (3.4), we have

$$y(q(x,t),t)=0, \qquad x<a \text{ } or \text{ } x>c.$$

Therefore, when $x>q(c,t)$, we obtain

$$\begin{aligned}u(x,t)&=G*y(x,t)\\&=\frac{1}{2}e^{-x}\int_{q(a,t)}^{q(c,t)}e^{\xi}y(\xi,t)d\xi\\&=\frac{1}{2}e^{-x}E(t),\end{aligned}$$

and when $x<q(a,t)$, we have

$$\begin{aligned}u(x,t)&=G*y(x,t)\\&=\frac{1}{2}e^{x}\int_{q(a,t)}^{q(c,t)}e^{-\xi}y(\xi,t)d\xi\\&=\frac{1}{2}e^{x}F(t).\end{aligned}$$

Now, we prove the monotonicity of $e^{\lambda t}E(t)$ and $e^{\lambda t}F(t)$.

(1) By (4.1), for $E(t)$, taking derivative with respect to x variable, we have

$$\frac{dE(t)}{dt}=\int_{\mathbb{R}}e^{\xi}y_t(\xi,t)d\xi.$$

From equation (1.1), we obtain

$$\begin{aligned}\frac{dE(t)}{dt}&=-\int_{\mathbb{R}}e^{\xi}(u^2y_\xi+buu_\xi y)d\xi-\lambda\int_{\mathbb{R}}e^{\xi}y(\xi,t)d\xi\\&=-\int_{\mathbb{R}}e^{\xi}(u^2y_\xi+buu_\xi y)d\xi-\lambda E(t).\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{d[e^{\lambda t}E(t)]}{dt}&=e^{\lambda t}[\lambda E(t)+\frac{dE(t)}{dt}]\\&=-e^{\lambda t}\int_{\mathbb{R}}e^{\xi}(u^2y_\xi+buu_\xi y)d\xi.\end{aligned}$$

Let

$$\begin{aligned}J_1&=-\int_{\mathbb{R}}e^{\xi}(u^2y_\xi+buu_\xi y)d\xi\\&=-\int_{\mathbb{R}}e^{\xi}u^2y_\xi d\xi-b\int_{\mathbb{R}}e^{\xi}yuu_\xi d\xi\\&=-e^{\xi}u^2y\Big|_{\mathbb{R}}+\int_{\mathbb{R}}ye^{\xi}u^2d\xi+2\int_{\mathbb{R}}ye^{\xi}uu_\xi d\xi-b\int_{\mathbb{R}}e^{\xi}yuu_\xi d\xi\\&=-e^{\xi}u^2y\Big|_{\mathbb{R}}+\int_{\mathbb{R}}ye^{\xi}u^2d\xi+(2-b)\int_{\mathbb{R}}e^{\xi}uu_\xi(u-u_{\xi\xi})d\xi\end{aligned}$$

$$= -e^{\xi}u^2y\Big|_{\mathbb{R}} + \int_{\mathbb{R}} e^{\xi}u^3 d\xi - \int_{\mathbb{R}} e^{\xi}u^2 u_{\xi\xi} d\xi$$
$$+ (2-b)\int_{\mathbb{R}} e^{\xi}u^2 u_{\xi} d\xi - (2-b)\int_{\mathbb{R}} e^{\xi} u u_{\xi} u_{\xi\xi} d\xi$$
$$= -e^{\xi}u^2y\Big|_{\mathbb{R}} - e^{\xi}u^2u_{\xi}\Big|_{\mathbb{R}} - \frac{2-b}{2}e^{\xi}uu_{\xi}^2\Big|_{\mathbb{R}} + \frac{3-b}{3}e^{\xi}u^3\Big|_{\mathbb{R}}$$
$$+ \frac{b}{3}\int_{\mathbb{R}} e^{\xi}u^3 d\xi + \frac{6-b}{2}\int_{\mathbb{R}} e^{\xi}uu_{\xi}^2 d\xi + \frac{2-b}{2}\int_{\mathbb{R}} e^{\xi}u_{\xi}^3 d\xi$$

and

$$\begin{aligned} H_1 &= -e^{\xi}u^2y\Big|_{\mathbb{R}} - e^{\xi}u^2u_{\xi}\Big|_{\mathbb{R}} - \frac{2-b}{2}e^{\xi}uu_{\xi}^2\Big|_{\mathbb{R}} + \frac{3-b}{3}e^{\xi}u^3\Big|_{\mathbb{R}} \\ &= -e^{\xi}(u^2y + u^2u_{\xi} + \frac{2-b}{2}uu_{\xi}^2 - \frac{3-b}{3}u^3)\Big|_{\mathbb{R}} \\ &= -e^{\xi}(\frac{b}{3}u^3 + \frac{2-b}{2}uu_{\xi}^2)\Big|_{\mathbb{R}} \\ &= -[\lim_{\xi\to+\infty} e^{\xi}(\frac{b}{3}u^3 + \frac{2-b}{2}uu_{\xi}^2) - \lim_{\xi\to-\infty} e^{\xi}(\frac{b}{3}u^3 + \frac{2-b}{2}uu_{\xi}^2)] \\ &= 0. \end{aligned}$$

Then, we can obtain

$$\frac{d[e^{\lambda t}E(t)]}{dt} = e^{\lambda t}(\frac{b}{3}\int_{\mathbb{R}} e^{\xi}u^3 d\xi + \frac{6-b}{2}\int_{\mathbb{R}} e^{\xi}uu_{\xi}^2 d\xi + \frac{2-b}{2}\int_{\mathbb{R}} e^{\xi}u_{\xi}^3 d\xi).$$

Since $y_0 \leqslant 0$ with compact support in an internal $[a, c]$, a direct consequence of (3.1) and (3.2) implies that

$$u(x,t) \leqslant 0, \quad u(x,t) + u_x(x,t) \leqslant 0, \quad u(x,t) - u_x(x,t) \leqslant 0. \tag{4.3}$$

For all $t \in [0, T]$ and $x \in R$, it follows that

$$u^3 + u_x^3 = (u+u_x)(u^2 - uu_x + u_x^2) = (u+u_x)[(u - \frac{1}{2}u_x)^2 + \frac{3}{4}u_x^2] \leqslant 0 \tag{4.4}$$

and

$$u^3 - u_x^3 = (u-u_x)(u^2 + uu_x + u_x^2) = (u-u_x)[(u + \frac{1}{2}u_x)^2 + \frac{3}{4}u_x^2] \leqslant 0. \tag{4.5}$$

If $0 \leqslant b \leqslant 2$, we have

$$\begin{aligned} \frac{d[e^{\lambda t}E(t)]}{dt} &= e^{\lambda t}(\frac{b}{3}\int_{\mathbb{R}} e^{\xi}u^3 d\xi + \frac{6-b}{2}\int_{\mathbb{R}} e^{\xi}uu_{\xi}^2 d\xi + \frac{2-b}{2}\int_{\mathbb{R}} e^{\xi}u_{\xi}^3 d\xi) \\ &< e^{\lambda t}(m_1\int_{\mathbb{R}} e^{\xi}(u^3 + u_{\xi}^3) d\xi + m_2\int_{\mathbb{R}} e^{\xi}u_{\xi}^2(u+u_{\xi}) d\xi) \\ &< 0, \end{aligned}$$

where $m_1, m_2 \geqslant 0$, and satisfy that $m_1 + m_2 = \frac{2-b}{2}$. If $2 < b \leqslant 6$, we have

$$\frac{d[e^{\lambda t}E(t)]}{dt} = e^{\lambda t}(\frac{b}{3}\int_{\mathbb{R}} e^{\xi}u^3 d\xi + \frac{6-b}{2}\int_{\mathbb{R}} e^{\xi}uu_{\xi}^2 d\xi + \frac{2-b}{2}\int_{\mathbb{R}} e^{\xi}u_{\xi}^3 d\xi)$$

$$
\begin{aligned}
&= e^{\lambda t}(\frac{b}{3}\int_{\mathbb{R}} e^{\xi}u^3 d\xi + \frac{6-b}{2}\int_{\mathbb{R}} e^{\xi}uu_\xi^2 d\xi + \frac{2-b}{2}\int_{\mathbb{R}} e^{\xi}u_\xi^3 d\xi)\\
&< e^{\lambda t}(m_3\int_{\mathbb{R}} e^{\xi}(u^3-u_\xi^3)d\xi + m_4\int_{\mathbb{R}} e^{\xi}u_\xi^2(u-u_\xi)d\xi)\\
&< 0,
\end{aligned}
$$

where $m_3, m_4 \geqslant 0$, and satisfy that $m_1 + m_2 = \frac{b-2}{2}$. Then, for any $x \in \mathbb{R}$, all $t \in [0, T]$ and $b \in [0, 6]$, we have

$$\frac{d[e^{\lambda t}E(t)]}{dt} < 0,$$

which is the $e^{\lambda t}E(t)$ is a strictly decreasing function, when $y_0 \leqslant 0$ and $b \in [0, 6]$.

Since $y_0 \geqslant 0$ with compact support in an internal $[a, c]$, a direct consequence of (3.1) and (3.2) implies that

$$u(x,t) \geqslant 0, \quad u(x,t) + u_x(x,t) \geqslant 0, \quad u(x,t) - u_x(x,t) \geqslant 0. \tag{4.6}$$

For all $t \in [0, T]$ and $x \in \mathbb{R}$, it follows that

$$u^3 + u_x^3 = (u + u_x)[(u - \frac{1}{2}u_x)^2 + \frac{3}{4}u_x^2] \geqslant 0 \tag{4.7}$$

and

$$u^3 - u_x^3 = (u - u_x)[(u + \frac{1}{2}u_x)^2 + \frac{3}{4}u_x^2] \geqslant 0. \tag{4.8}$$

If $0 \leqslant b \leqslant 2$, we have

$$
\begin{aligned}
\frac{d[e^{\lambda t}E(t)]}{dt} &> e^{\lambda t}(\tilde{m_1}\int_{\mathbb{R}} e^{\xi}(u^3+u_\xi^3)d\xi + \tilde{m_2}\int_{\mathbb{R}} e^{\xi}u_\xi^2(u+u_\xi)d\xi)\\
&> 0,
\end{aligned}
$$

where $\tilde{m_1}, \tilde{m_2} > 0$, satisfying that $\tilde{m_1} + \tilde{m_2} = \frac{2-b}{2}$. If $2 < b \leqslant 6$, we have

$$
\begin{aligned}
\frac{d[e^{\lambda t}E(t)]}{dt} &> e^{\lambda t}(\tilde{m_3}\int_{\mathbb{R}} e^{\xi}(u^3-u_\xi^3)d\xi + \tilde{m_4}\int_{\mathbb{R}} e^{\xi}u_\xi^2(u-u_\xi)d\xi)\\
&> 0,
\end{aligned}
$$

where $\tilde{m_3}, \tilde{m_4} > 0$, satisfy that $\tilde{m_3} + \tilde{m_4} = \frac{b-2}{2}$.
Then, for any $x \in \mathbb{R}$, all $t \in [0, T]$ and $b \in [0, 6]$, we have

$$\frac{d[e^{\lambda t}E(t)]}{dt} > 0.$$

Therefore, the conclusion is right, $e^{\lambda t}E(t)$ is a strictly increasing function, when $y_0 \leqslant 0$ and $b \in [0, 6]$.

(2) For $F(t)$, similar as the method of case (1), we have

$$\frac{dF(t)}{dt} = \int_{\mathbb{R}} e^{-\xi}y_t(\xi,t)d\xi$$

$$= -\int_{\mathbb{R}} e^{-\xi}(u^2 y_\xi + buu_\xi y)d\xi - \lambda F(t)$$

and

$$\frac{d[e^{\lambda t}F(t)]}{dt} = -e^{\lambda t}\int_{\mathbb{R}} e^{-\xi}(u^2 y_\xi + buu_\xi y)d\xi = -e^{-\lambda t}J_2.$$

For J_2, we have

$$\begin{aligned}
J_2 &= -\int_{\mathbb{R}} e^{-\xi}(u^2 y_\xi + buu_\xi y)d\xi \\
&= -\int_{\mathbb{R}} e^{-\xi}u^2 y_\xi d\xi - b\int_{\mathbb{R}} e^{-\xi} yuu_\xi d\xi \\
&= -e^{-\xi}u^2 y\Big|_{\mathbb{R}} - \int_{\mathbb{R}} ye^{-\xi}u^2 d\xi + 2\int_{\mathbb{R}} ye^{-\xi}uu_\xi d\xi - b\int_{\mathbb{R}} e^{-\xi}yuu_\xi d\xi \\
&= -e^{-\xi}u^2 y\Big|_{\mathbb{R}} + \int_{\mathbb{R}} ye^{-\xi}u^2 d\xi + (2-b)\int_{\mathbb{R}} e^{-\xi}uu_\xi(u - u_{\xi\xi})d\xi \\
&= -e^{-\xi}u^2 y\Big|_{\mathbb{R}} - \int_{\mathbb{R}} e^{-\xi}u^3 d\xi + \int_{\mathbb{R}} e^{-\xi}u^2 u_{\xi\xi} d\xi \\
&\quad + (2-b)\int_{\mathbb{R}} e^{-\xi}u^2 u_\xi d\xi - (2-b)\int_{\mathbb{R}} e^{-\xi}uu_\xi u_{\xi\xi} d\xi \\
&= -e^{-\xi}u^2 y\Big|_{\mathbb{R}} + e^{-\xi}u^2 u_\xi\Big|_{\mathbb{R}} - \frac{2-b}{2}e^{-\xi}uu_\xi^2\Big|_{\mathbb{R}} + \frac{3-b}{3}e^{-\xi}u^3\Big|_{\mathbb{R}} \\
&\quad - \frac{b}{3}\int_{\mathbb{R}} e^{-\xi}u^3 d\xi - \frac{6-b}{2}\int_{\mathbb{R}} e^{-\xi}uu_\xi^2 d\xi - \frac{2-b}{2}\int_{\mathbb{R}} e^{-\xi}u_\xi^3 d\xi \\
&= -\frac{b}{3}\int_{\mathbb{R}} e^{-\xi}u^3 d\xi - \frac{6-b}{2}\int_{\mathbb{R}} e^{-\xi}uu_\xi^2 d\xi - \frac{2-b}{2}\int_{\mathbb{R}} e^{-\xi}u_\xi^3 d\xi.
\end{aligned}$$

Considering $y_0 \leqslant 0$, from (4.3), (4.4) and (4.5), if $0 \leqslant b \leqslant 2$, we have

$$\begin{aligned}
\frac{d[e^{\lambda t}F(t)]}{dt} &> e^{\lambda t}(n_1\int_{\mathbb{R}} e^{-\xi}(u_\xi^3 - u^3)d\xi + n_2\int_{\mathbb{R}} e^{-\xi}u_\xi^2(u_\xi - u)d\xi) \\
&> 0,
\end{aligned}$$

where $n_1, n_2 \geqslant 0$ and $n_1 + n_2 = \frac{2-b}{2}$. If $2 < b \leqslant 6$, we obtain

$$\begin{aligned}
\frac{d[e^{\lambda t}F(t)]}{dt} &> -e^{\lambda t}(n_3\int_{\mathbb{R}} e^{-\xi}(u^3 + u_\xi^3)d\xi + n_4\int_{\mathbb{R}} e^{-\xi}u_\xi^2(u + u_\xi)d\xi) \\
&> 0,
\end{aligned}$$

where $n_3, n_4 \geqslant 0$ and $n_3 + n_4 = \frac{b-2}{2}$. Therefore, $e^{\lambda t}F(t)$ is a strictly increasing function, when $y_0 \leqslant 0$ and $b \in [0, 6]$ for any $x \in \mathbb{R}$ and all $t \in [0, T]$.

On the other hand, when $y_0 \geqslant 0$, from (4.6), (4.7) and (4.8), if $0 \leqslant b \leqslant 2$, we have

$$\begin{aligned}
\frac{d[e^{\lambda t}F(t)]}{dt} &< e^{\lambda t}(\tilde{n_1}\int_{\mathbb{R}} e^{-\xi}(u_\xi^3 - u^3)d\xi + \tilde{n_2}\int_{\mathbb{R}} e^{-\xi}u_\xi^2(u_\xi - u)d\xi) \\
&< 0,
\end{aligned}$$

where $\tilde{n_1}, \tilde{n_2} \geqslant 0$ and $\tilde{n_1} + \tilde{n_2} = \frac{2-b}{2}$. Further, if $2 < b \leqslant 6$, we obtain

$$\begin{aligned}\frac{d[e^{\lambda t}F(t)]}{dt} &< -e^{\lambda t}(\tilde{n_3}\int_{\mathbb{R}} e^{-\xi}(u^3 + u_\xi^3)d\xi + \tilde{n_4}\int_{\mathbb{R}} e^{-\xi}u_\xi^2(u + u_\xi)d\xi)\\ &< 0,\end{aligned}$$

where $\tilde{n_3}, \tilde{n_4} \geqslant 0$ and $\tilde{n_3} + \tilde{n_4} = \frac{b-2}{2}$. Therefore, for any $x \in \mathbb{R}$ and all $t \in [0, T]$, when $b \in [0, 6]$, $e^{\lambda t}F(t)$ is a decreasing function as $y_0 \geqslant 0$. Therefore, the conclusion is established based on the calculation above. □

5. Large time behavior for the support of momentum density

In this section, we will discuss the large time behavior for the support of momentum density.

Lemma 5.1. *If $b \geqslant 0$, $\lambda \leqslant 0$, assume the initial value $u_0 \not\equiv 0$ has a compact supported set $[a, c]$, say $y_0 \equiv 0$, if $x < a$ or $x > c$. Then, if $y_0(x)(\not\equiv 0)$ does not change sign, $x \in [a, c]$, we have*

$$\lim_{t\to+\infty} e^{\lambda t}F(t) = 0.$$

Proof. When $y_0(x)(\not\equiv 0)$ does not change sign, $x \in [a, c]$, assume that

$$\lim_{t\to+\infty} e^{\lambda t}F(t) \neq 0.$$

Therefore, there is a constant $\varepsilon_0 > 0$, for any $T > 0$, there will exist a $t > T$ such that $|e^{\lambda t}F(t)| \geqslant \varepsilon_0$. Then, by calculation, we have

$$\begin{aligned}\frac{dq(a,t)}{dt} &= u^2(q(a,t),t)\\ &= (\frac{1}{2}e^{q(a,t)}F(t))^2\\ &= \frac{1}{4}e^{2q(a,t)}\frac{(e^{\lambda t}F(t))^2}{e^{2\lambda t}}\\ &\geqslant \frac{1}{4}e^{2q(a,t)}\frac{{\varepsilon_0}^2}{e^{2\lambda t}}.\end{aligned}$$

It follows that

$$e^{-2q(a,t)} \leqslant \frac{{\varepsilon_0}^2}{4\lambda}(e^{-2\lambda t} - 1) + e^{-2c}.$$

Taking $T = -\frac{\ln(1-4\lambda e^{-2c}/{\varepsilon_0}^2)}{2\lambda}$, for any $t > T$, we obtain

$$\frac{{\varepsilon_0}^2}{4\lambda}(e^{-2\lambda t} - 1) + e^{-2c} \leqslant 0.$$

This is the contradiction. Therefore, we have

$$\lim_{t\to+\infty} e^{\lambda t}F(t) = 0.$$

□

Theorem 5.1. *If $b > 2, \lambda \leqslant 0$, suppose that $y_0(x) \in L_{\frac{2}{b}}$ has a compact supported set $[a, c]$. Then, if $y_0(x)(\not\equiv 0)$ does not change sign, $x \in [a, c]$, we have*

$$e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}} \to +\infty, \quad as \quad t \to +\infty.$$

Proof. (1) When $y_0 \geqslant 0(\not\equiv 0), x \in [a, c]$, for any $t \geqslant 0$, we have $F(t) > 0$. By direct calculation, we have

$$\begin{aligned}
e^{-\lambda t}[\int_a^c (y_0)^{\frac{2}{b}} dx]^{\frac{b}{2}} &= [\int_a^c (y_0 e^{-\lambda t})^{\frac{2}{b}} dx]^{\frac{b}{2}} \\
&= [\int_a^c (y(q,t) q_x^{\frac{b}{2}})^{\frac{2}{b}} dx]^{\frac{b}{2}} \\
&= [\int_a^c (y(q,t))^{\frac{2}{b}} q_x dx]^{\frac{b}{2}} \\
&= [\int_{q(a,t)}^{q(c,t)} (y(\xi,t))^{\frac{2}{b}} d\xi]^{\frac{b}{2}} \\
&\leqslant (\int_{q(a,t)}^{q(c,t)} y(\xi,t) e^{-\xi} d\xi)(\int_{q(a,t)}^{q(c,t)} e^{\frac{2\xi}{b-2}} d\xi)^{\frac{b-2}{2}} \\
&= F(t)[(\frac{b-2}{2})(e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}})]^{\frac{b-2}{2}}.
\end{aligned}$$

Therefore, we obtain

$$e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}} \geqslant (\frac{2}{b-2})[\frac{(\int_a^c y_0^{\frac{2}{b}} dx)^{\frac{b}{2}}}{F(t)e^{\lambda t}}]^{\frac{2}{b-2}}.$$

According to limit that $\lim_{t \to +\infty} e^{\lambda t} F(t) = 0$, we have

$$e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}} \to +\infty, \quad as \quad t \to +\infty.$$

(2) When $y_0 \leqslant 0(\not\equiv 0), x \in [a, c]$, for any $t \geqslant 0$, we have $F(t) < 0$. By direct calculation, we have

$$\begin{aligned}
e^{-\lambda t}[\int_a^c (-y_0)^{\frac{2}{b}} dx]^{\frac{b}{2}} &= [\int_a^c (-y_0 e^{-\lambda t})^{\frac{2}{b}} dx]^{\frac{b}{2}} \\
&= [\int_a^c (-y(q,t) q_x^{\frac{b}{2}})^{\frac{2}{b}} dx]^{\frac{b}{2}} \\
&= [\int_a^c (-y(q,t))^{\frac{2}{b}} q_x dx]^{\frac{b}{2}} \\
&= [\int_{q(a,t)}^{q(c,t)} (-y(\xi,t))^{\frac{2}{b}} d\xi]^{\frac{b}{2}} \\
&\leqslant (\int_{q(a,t)}^{q(c,t)} -y(\xi,t) e^{-\xi} d\xi)(\int_{q(a,t)}^{q(c,t)} e^{\frac{2\xi}{b-2}} d\xi)^{\frac{b-2}{2}} \\
&= -F(t)[(\frac{b-2}{2})(e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}})]^{\frac{b-2}{2}}.
\end{aligned}$$

Hence, we obtain

$$e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}} \geqslant (\frac{2}{b-2})[\frac{(\int_a^c (-y_0)^{\frac{2}{b}} dx)^{\frac{b}{2}}}{-F(t)e^{\lambda t}}]^{\frac{2}{b-2}}.$$

Therefore, we finally have

$$e^{\frac{2q(c,t)}{b-2}} - e^{\frac{2q(a,t)}{b-2}} \to +\infty, \quad as \quad t \to +\infty.$$

□

Theorem 5.2. *If $b = 2, \lambda \leqslant 0$, suppose that $y_0(x) \in L_1$ has a compact supported set $[a, c]$, then if $y_0(x)(\not\equiv 0)$ does not change sign, $x \in [a, c]$, we have*

$$q(c,t) \to +\infty, \quad as \quad t \to +\infty.$$

Proof. (1) When $y_0 \geqslant 0(\not\equiv 0), x \in [a, c]$, for any $t \geqslant 0$, we have $F(t) > 0$. Therefore, by direct calculation, we obtain

$$\begin{aligned} e^{-\lambda t} \int_a^c y_0 dx &= \int_a^c y_0 e^{-\lambda t} dx \\ &= \int_a^c y(q,t) q_x dx \\ &\leqslant e^{q(c,t)} \int_{q(a,t)}^{q(c,t)} y(\xi,t) e^{-\xi} d\xi \\ &= e^{q(c,t)} F(t). \end{aligned}$$

Then, according to the limit that $\lim\limits_{t\to+\infty} e^{\lambda t} F(t) = 0$, we have

$$e^{q(c,t)} \geqslant \frac{\int_a^c y_0 dx}{F(t)e^{\lambda t}} \to +\infty, \quad as \quad t \to +\infty,$$

which is

$$q(c,t) \to +\infty, \quad as \quad t \to +\infty.$$

(2) When $y_0 \leqslant 0(\not\equiv 0), x \in [a, c]$, for any $t \geqslant 0$, we have $F(t) < 0$. Therefore, by direct calculation, we obtain

$$\begin{aligned} e^{-\lambda t} \int_a^c -y_0 dx &= \int_a^c -y_0 e^{-\lambda t} dx \\ &= \int_a^c -y(q,t) q_x dx \\ &\leqslant e^{q(c,t)} \int_{q(a,t)}^{q(c,t)} -y(\xi,t) e^{-\xi} d\xi \\ &= -e^{q(c,t)} F(t). \end{aligned}$$

Then, we have

$$e^{q(c,t)} \geqslant \frac{\int_a^c -y_0 dx}{-F(t)e^{\lambda t}} \to +\infty, \quad as \quad t \to +\infty,$$

which is

$$q(c,t) \to +\infty, \quad as \quad t \to +\infty.$$

□

Theorem 5.3. *Let $\lambda \leqslant 0$, when $0 < b < 2$, $y_0(x) \in L_{\frac{2}{b}}$ or $b = 0$, $y_0(x) \in L_\infty$, assume that $y_0(x) \in L_{\frac{2}{b}}$ has a compact supported set $[a,c]$, then*

(1) if $y_0(x) \geqslant 0 (\not\equiv 0)$, $x \in [a,c]$, we have

$$e^{-4\int_0^t \inf_{x\in\mathbb{R}}(uu_x)ds}(e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}}) \to +\infty, \quad as \quad t \to +\infty.$$

(2) if $y_0(x) \leqslant 0 (\not\equiv 0)$, $x \in [a,c]$, we have

$$e^{-4\int_0^t \sup_{x\in\mathbb{R}}(uu_x)ds}(e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}}) \to +\infty, \quad as \quad t \to +\infty.$$

Proof. (1) When $y_0 \geqslant 0$ and $\lambda \leqslant 0$, if $x \in [a,c]$ and $0 < b < 2$, we obtain

$$\begin{aligned}\frac{d}{dt}\|y\|_{L^1} &= (2-b)\int_{\mathbb{R}} yuu_x dx - \lambda \|y\|_{L^1} \\ &\geqslant [(2-b)\inf_{x\in\mathbb{R}}(uu_x) - \lambda]\|y\|_{L^1},\end{aligned}$$

which is

$$\begin{aligned}\|y\|_{L^1} &\geqslant e^{\int_0^t [(2-b)\inf_{x\in\mathbb{R}}(uu_x)-\lambda]ds}\|y_0\|_{L^1} \\ &= e^{\int_0^t (2-b)\inf_{x\in\mathbb{R}}(uu_x)ds - \lambda t}\|y_0\|_{L^1}.\end{aligned}$$

From equation (3.5), we have

$$\begin{aligned}\|y\|_{L^1} &= \int_{\mathbb{R}} y dx \\ &= \int_{\mathbb{R}} y^{\frac{1}{2}}(ye^{-\xi})^{\frac{1}{2}} e^{\frac{\xi}{2}} d\xi \\ &\leqslant (\int_{\mathbb{R}} y^{\frac{2}{b}} d\xi)^{\frac{b}{4}} (\int_{\mathbb{R}} ye^{-\xi} d\xi)^{\frac{1}{2}} (\int_{q(a,t)}^{q(c,t)} e^{\frac{2\xi}{2-b}} d\xi)^{\frac{2-b}{4}} \\ &= (\int_{\mathbb{R}} y_0^{\frac{2}{b}} dx)^{\frac{b}{4}} (e^{-\lambda t}F(t))^{\frac{1}{2}} [\frac{2-b}{2}(e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}})]^{\frac{2-b}{4}}.\end{aligned}$$

Then, we obtain

$$e^{-4\int_0^t \inf_{x\in\mathbb{R}}(uu_x)ds}(e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}}) \geqslant \frac{2}{2-b}[\frac{\|y_0\|_{L^1}}{\|y_0\|_{L^{\frac{2}{b}}}^{\frac{1}{2}}(e^{\lambda t}F(t))^{\frac{1}{2}}}]^{\frac{4}{2-b}} \to +\infty, \quad as \quad t \to +\infty.$$

If $b = 0$, we have

$$\|y\|_{L^1} \geqslant e^{2\int_0^t \inf_{x\in\mathbb{R}}(uu_x)ds - \lambda t}\|y_0\|_{L^1}$$

and

$$\begin{aligned}\|y\|_{L^1} &= \int_{\mathbb{R}} y dx \\ &= \int_{\mathbb{R}} y^{\frac{1}{2}} (ye^{-\xi})^{\frac{1}{2}} e^{\frac{\xi}{2}} d\xi \\ &\leqslant \lim_{b\to 0} (\int_{\mathbb{R}} y^{\frac{2}{b}} d\xi)^{\frac{b}{4}} (\int_{\mathbb{R}} ye^{-\xi} d\xi)^{\frac{1}{2}} (\int_{q(a,t)}^{q(c,t)} e^{\frac{2\xi}{2-b}} d\xi)^{\frac{2-b}{4}} \\ &= \lim_{b\to 0} [(\int_{\mathbb{R}} y_0^{\frac{2}{b}} dx)^{\frac{b}{4}} (e^{-\lambda t} F(t))^{\frac{1}{2}} [\frac{2-b}{2} (e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}})]^{\frac{2-b}{4}}].\end{aligned}$$

Finally, we obtain

$$e^{-4\int_0^t \inf_{x\in\mathbb{R}} (uu_x) ds} (e^{q(c,t)} - e^{q(a,t)}) \geqslant [\frac{\|y_0\|_{L^1}}{(\lim_{b\to 0} \|y_0\|^{\frac{1}{2}}_{L^{\frac{2}{b}}})(e^{\lambda t}F(t))^{\frac{1}{2}}}]^2 \to +\infty, \quad as \quad t \to +\infty.$$

(2) When $y_0 \leqslant 0$, if $x \in [a, c]$ and $0 < b < 2$, we obtain

$$\frac{d}{dt} \|y\|_{L^1} \leqslant [(2-b) \sup_{x\in\mathbb{R}} (uu_x) - \lambda] \|y\|_{L^1},$$

which is

$$\begin{aligned}\|-y\|_{L^1} &\geqslant e^{\int_0^t [(2-b)\sup_{x\in\mathbb{R}}(uu_x) - \lambda] ds} \|-y_0\|_{L^1} \\ &= e^{\int_0^t (2-b)\sup_{x\in\mathbb{R}}(uu_x) ds - \lambda t} \|-y_0\|_{L^1},\end{aligned}$$

and we have

$$\begin{aligned}\|-y\|_{L^1} &= \int_{\mathbb{R}} -y dx \\ &= \int_{\mathbb{R}} (-y)^{\frac{1}{2}} (-ye^{-\xi})^{\frac{1}{2}} e^{\frac{\xi}{2}} d\xi \\ &\leqslant (\int_{\mathbb{R}} (-y)^{\frac{2}{b}} d\xi)^{\frac{b}{4}} (\int_{\mathbb{R}} -ye^{-\xi} d\xi)^{\frac{1}{2}} (\int_{q(a,t)}^{q(c,t)} e^{\frac{2\xi}{2-b}} d\xi)^{\frac{2-b}{4}} \\ &= (\int_{\mathbb{R}} (-y_0)^{\frac{2}{b}} dx)^{\frac{b}{4}} (-e^{-\lambda t} F(t))^{\frac{1}{2}} [\frac{2-b}{2} (e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}})]^{\frac{2-b}{4}}.\end{aligned}$$

Then, we get

$$e^{-4\int_0^t \sup_{x\in\mathbb{R}} (uu_x) ds} (e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}}) \geqslant \frac{2}{2-b} [\frac{\|y_0\|_{L^1}}{\|y_0\|^{\frac{1}{2}}_{L^{\frac{2}{b}}} (e^{\lambda t}F(t))^{\frac{1}{2}}}]^{\frac{4}{2-b}} \to +\infty, as \quad t \to +\infty.$$

If $b = 0$, we have

$$\|-y\|_{L^1} \geqslant e^{2\int_0^t \sup_{x\in\mathbb{R}} (uu_x) ds - \lambda t} \|-y_0\|_{L^1}$$

and

$$\|-y\|_{L^1} = \int_{\mathbb{R}} -y dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} (-y)^{\frac{1}{2}}(-ye^{-\xi})^{\frac{1}{2}} e^{\frac{\xi}{2}} d\xi \\
&\leqslant \lim_{b\to 0}\Big(\int_{\mathbb{R}} (-y)^{\frac{2}{b}} d\xi\Big)^{\frac{b}{4}}\Big(\int_{\mathbb{R}} (-y)e^{-\xi} d\xi\Big)^{\frac{1}{2}}\Big(\int_{q(a,t)}^{q(c,t)} e^{\frac{2\xi}{2-b}} d\xi\Big)^{\frac{2-b}{4}} \\
&= \lim_{b\to 0}\Big[\Big(\int_{\mathbb{R}} (-y_0)^{\frac{2}{b}} dx\Big)^{\frac{b}{4}}(-e^{-\lambda t}F(t))^{\frac{1}{2}}\Big[\frac{2-b}{2}\big(e^{\frac{2q(c,t)}{2-b}} - e^{\frac{2q(a,t)}{2-b}}\big)\Big]^{\frac{2-b}{4}}\Big].
\end{aligned}
$$

Hence, we obtain

$$
e^{-4\int_0^t \sup\limits_{x\in\mathbb{R}}(uu_x)ds}\big(e^{q(c,t)} - e^{q(a,t)}\big) \geqslant \Big[\frac{\|y_0\|_{L^1}}{(\lim\limits_{b\to 0}\|y_0\|^{\frac{1}{2}}_{L^{\frac{2}{b}}})(e^{\lambda t}F(t))^{\frac{1}{2}}}\Big]^2 \to +\infty, \quad as \quad t \to +\infty.
$$

Therefore, the conclusion is established. □

6. Persistence property

In this section, we will consider the persistence property in Sobolev space.

Definition 6.1. A nonnegative function $v : \mathbb{R}^n \to \mathbb{R}$ is called sub-multiplicative, if $v(x+y) \leq v(x)v(y)$ holds for all $x, y \in \mathbb{R}^n$. Given a sub-multiplicative function v, a positive function ϕ is called v-moderate, if there exists a constant $C > 0$ such that $\phi(x+y) \leq Cv(x)\phi(y)$ holds for all $x, y \in \mathbb{R}^n$.

It is proved in Brandolese [46] that ϕ is v-moderate, if and only if the weighted Young inequality

$$
\|(f_1 * f_2)\phi\|_{L^p} \leq C\|f_1 v\|_{L^1}\|f_2\phi\|_{L^p} \tag{6.1}
$$

holds for any two measurable functions f_1, f_2 and $1 \leq p \leq \infty$.

Definition 6.2. We say that $\phi : \mathbb{R} \to (0, +\infty)$ is an admissible weight for (1.1), if it is locally absolutely continuous such that $|\phi^{'}(x)| \leq A|\phi(x)|$ for some $A > 0$ and a.e. $x \in \mathbb{R}$, and is v-moderate with a sub-multiplicative function v satisfying $\inf_R v > 0$ and

$$
\int_R v(x)e^{-|x|}dx < \infty. \tag{6.2}
$$

Theorem 6.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$, $\lambda \in \mathbb{R}$, $2 \leq p \leq \infty$ and $u \in C([0,t); H^s(\mathbb{R})) \cap C^1([0,t); H^{s-1}(\mathbb{R}))$ be the strong solution to (1.1) starting from u_0 such that $\phi u_0, \phi u_{0x} \in L^p(\mathbb{R})$ for an admissible weight function ϕ of (1.1). Then, for all $t \in [0,T)$, we have the estimate*

$$
\|\phi u(.,t)\|_{L^\infty} + \|\phi u_x(.,t)\|_{L^\infty} \leq e^{C(M+|\lambda|)t}(\|\phi u_0(.,t)\|_{L^\infty} + \|\phi u_{0x}(.,t)\|_{L^\infty}),
$$

where the constant C depends only on the weight v, ϕ and

$$
M = \sup_{t\in[0,T)} \|u\|_{H^s}.
$$

Remark 6.1. The example for admissible weight functions can be found in [18]

$$\phi(x) = \phi_{\alpha,\beta,\gamma,\delta}(x) = e^{\alpha|x|^{\beta}}(1+|x|)^{\gamma}\log(e+|x|^{\delta}, \tag{6.3}$$

where we require that $\alpha \geq 0, \quad 0 \leq \beta \leq 1, \quad \alpha\beta < 1.$

Proof. For the sake of convenience, we rewrite (1.1) as a transport equation (1.5) with

$$F(u) = [(6-b)uu_x u_{xx} + 2u_x^3 + bu^2 u_x],$$

where $G(x) = e^{-\frac{|x|}{2}}$ is again the Green's function of the operator $(1-\partial_x^2)$. For any $N \in \mathbb{N}\backslash\{0\}$, we define N-truncation:

$$\phi_N(x) = min\{\phi(x), N\}.$$

Then, it is easy to check that $\phi_N(x) : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function satisfying $\|\phi_N(x)\|_{L^\infty} \leq N$ and $|\phi_N^{'}| \leq A\phi_N(x)$ a.e. on R. For $p \in [2, +\infty)$, multiplying (1.5) by $\phi_N|\phi_N u|^{p-2}\phi_N u$ and integrating both sides on the lines, one can get

$$\|\phi_N u\|_{L^p}^{p-1}\frac{d}{dt}\|\phi_N u\|_{L^p} + \int_R u|\phi_N u|^p u_x dx + \int_R \phi_N G * F(u)|\phi_N u|^{p-2}\phi_N u dx$$
$$+ \lambda\int_{\mathbb{R}} u\phi_N|\phi_N u|^{p-2}\phi_N u dx = 0.$$

We observe that

$$|\int_{\mathbb{R}} u|\phi_N u|^p u_x dx| \leq C_1[\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2]\|\phi_N u\|_{L^p}^p,$$

and by Hölder's inequality that

$$|\int_{\mathbb{R}} \phi_N G * F(u)|\phi_N u|^{p-2}\phi_N u dx| \leq \|\phi_N G * F(u)\|_{L^p}\|\phi_N u\|_{L^p}^{p-1}.$$

Moreover, by using (6.1) and (6.2), we have

$$\|\phi_N G * F(u)\|_{L^p} \leq \|Gv\|_{L^1}\|\phi_N F(u)\|_{L^p} \leq \|\phi_N F(u)\|_{L^p}.$$

Besides, we have

$$|\lambda\int_{\mathbb{R}} u\phi_N|\phi_N u|^{p-2}\phi_N u dx| = |\lambda|\,\|\phi_N u\|_{L^p}^p.$$

Hence, we obtain

$$\frac{d}{dt}\|\phi_N u\|_{L^p} \leq C_2(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2 + |\lambda|)\|u\phi_N\|_{L^p} + \|\phi_N F(u)\|_{L^p}. \tag{6.4}$$

Differentiating (1.5) with respect to the variable x produces the following equation

$$u_{tx} + 2uu_x^2 + \partial_x(G * F(u)) + u^2 u_{xx} + \lambda u_x = 0.$$

Multiplying the above equation by $\phi_N|\phi_N u_x|^{p-2}\phi_N u_x$ and integrating over the line, one has

$$\begin{aligned}\|\phi_N u_x\|_{L^p}^{p-1}\frac{d}{dt}\|\phi_N u_x\|_{L^p} + \int_{\mathbb{R}} \phi_N\partial_x(G*F(u))|\phi_N u_x|^{p-2}\phi_N u_x dx \\ + \int_{\mathbb{R}} u^2 u_{xx}\phi_N|\phi_N u_x|^{p-2}\phi_N u_x dx \\ + 2\int_{\mathbb{R}} u u_x^2\phi_N|\phi_N u_x|^{p-2}\phi_N u_x dx \\ + \lambda\int_{\mathbb{R}} u_x\phi_N|\phi_N u|^{p-2}\phi_N u dx = 0,\end{aligned}$$

and also

$$\begin{aligned}\int_{\mathbb{R}} u^2 u_{xx}\phi_N|\phi_N u_x|^{p-2}\phi_N u_x dx &= \int_{\mathbb{R}} u^2|\phi_N u_x|^{p-2}\phi_N u_x[(u_x\phi_N)_x - u_x\partial_x\phi_N]dx \\ &= \int_{\mathbb{R}} u^2\partial_x(\frac{|\phi_N u_x|^p}{p})dx- \\ &= \int_{\mathbb{R}} u^2|\phi_N u_x|^{p-2}\phi_N u_x(u_x\phi_N^{'})dx.\end{aligned}$$

Note that since $|\phi_N^{'}(x)| \leq A\phi_N(x)$ a.e. on R, it follows that

$$|\int_{\mathbb{R}} u^2 u_{xx}\phi_N|\phi_N u_x|^{p-2}\phi_N u_x dx| \leq C_3(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2)(1+A)\|\phi_N u_x\|_{L^p}^p,$$

and

$$|\int_{\mathbb{R}} u u_x^2\phi_N|\phi_N u_x|^{p-2}\phi_N u_x dx| \leq C_4(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2)\|\phi_N u_x\|_{L^p}^p.$$

Then,

$$|\int_{\mathbb{R}} \phi_N\partial_x(G*F(u))|\phi_N u_x|^{p-2}\phi_N u_x dx| \leq \|\phi_N\partial_x(G*F(u))\|_{L^p}\|\phi_N u_x\|_{L^p}^{p-1}.$$

By using the fact $\partial_x G = -\frac{1}{2}sgn(x)e^{-|x|}$ in the weak sense and applying (6.1) and (6.2) again, we have

$$\|\phi_N\partial_x(G*F(u))\|_{L^p} \leq \|(\partial_x G)v\|_{L^1}\|\phi_N F(u)\|_{L^p} \leq \|\phi_N F(u)\|_{L^p}.$$

Besides, we have

$$|\lambda\int_{\mathbb{R}} u_x\phi_N|\phi_N u|^{p-2}\phi_N u dx| = |\lambda|\,\|\phi_N u_x\|_{L^p}^p.$$

Thus, we get

$$\frac{d}{dt}\|u_x\phi_N\|_{L^p} \leq C_5(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2 + |\lambda|)\|\phi_N u_x\|_{L^p} + \|\phi_N F(u)\|_{L^p}.$$

Combining above results together,

$$\frac{d}{dt}(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p})\le C_6(\|u\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2+|\lambda|)$$
$$(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p})+\|\phi_N F(u)\|_{L^p}. \qquad (6.5)$$

Further, we can easily conclude by using the definition of $F(u)$ that

$$\|\phi_N F(u)\|_{L^p}\le C_7(\|u\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2+\|u_{xx}\|_{L^\infty}^2+|\lambda|)(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p}).$$

Combining with (6.5), we obtain

$$\begin{aligned}\frac{d}{dt}(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p})\le{}& C(\|u\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2+\|u_{xx}\|_{L^\infty}^2+|\lambda|)\\ &(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p})\\ \le{}& C(M+|\lambda|)(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p}).\end{aligned}$$

By Gronwalls' inequality, we have

$$(\|u\phi_N\|_{L^p}+\|u_x\phi_N\|_{L^p})\le e^{C(M+|\lambda|)t}(\|u_0\phi_N\|_{L^p}+\|u_{0x}\phi_N\|_{L^p}).$$

Letting $p\to+\infty$, due to the term $e^{C(M+|\lambda|)t}$ is independent on p, which implies that

$$(\|u\phi_N\|_{L^\infty}+\|u_x\phi_N\|_{L^\infty})\le e^{C(M+|\lambda|)t}(\|u_0\phi_N\|_{L^\infty}+\|u_{0x}\phi_N\|_{L^\infty}).$$

This completes the proof. □

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