Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 5, October 2025, 1886–1904 DOI:10.12150/jnma.2025.1886

Existence, Uniqueness and Fourth-Order Numerical Method for Solving Fully Third-Order Nonlinear ODE with Integral Boundary Conditions*

Dang Quang A[1] and Dang Quang Long[2,†]

Abstract In this paper, we establish the existence, uniqueness and construct fourth-order numerical method for solving fully third-order nonlinear differential equation with integral boundary conditions. The method is based on the discretization of an iterative method on continuous level with the use of the trapezoidal quadrature formulas with corrections. Some examples demonstrate the applicability of the theoretical results of existence and uniqueness of solution and the fourth-order convergence of the proposed numerical method. The approach used for the third-order nonlinear differential equation with integral boundary conditions can be applied to differential equations of any order.

Keywords Third-order nonlinear differential equation, integral boundary conditions, iterative method, fourth-order convergence

MSC(2010) 34B15, 34L30, 65L10, 65L12.

1. Introduction

Third-order differential equations arise in many different fields of mechanics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [9]. Recently, third-order two-point or multipoint boundary value problems (BVPs) have attracted a lot of attention (see e.g., [1, 2, 4, 8, 14, 19] and references therein). It is known that BVPs with integral boundary conditions cover multipoint BVPs as special cases. It is worth mentioning here some works on third-order BVPs with integral boundary conditions [3, 10–12, 21–23].

In this paper we consider the following boundary value problem (BVP)

$$u'''(t) = f(t, u(t), u'(t), u''(t)),\ t \in (0, 1), \tag{1.1}$$

$$u(0) = c_1, u''(0) = c_2, u(1) = \int_0^1 g(s)u(s)ds + c_3, \tag{1.2}$$

† the corresponding author.
Email address: dquanga@gmail.com (Q.A Dang), dqlong88@gmail.com (Q.L. Dang)
1 Thai Nguyen University of Information and Communication Technology, Thai Nguyen, Vietnam
2 Institute of Information Technology, VAST, Hanoi, Vietnam
* The second author is funded by the Institute of Information Technology, VAST, under the grant number CS24.03.

where f, g are continuous functions and c_1, c_2, c_3 are real numbers. Some simplified versions of the above problem were studied in [11, 12, 22]. Namely, in [22] Zhao et al. investigated the existence, nonexistence, and multiplicity of positive solutions for the problem

$$u'''(t) = f(t, u(t)),\ t \in (0,1), \tag{1.3}$$

$$u(0) = 0, u''(0) = 0, u(1) = \int_0^1 g(s)u(s)ds \tag{1.4}$$

in ordered Banach spaces by means of fixed-point principle in cone and the fixed-point index theory for strict set contraction operator. One example was given to illustrate the existence of positive solutions although this solution was not shown. Later, in 2012 and 2013 Guo et al. [11, 12] by using the fixed-point index theory in a cone and nonlocal Green function obtained the existence of at least one positive solution for the problem of the type (1.3)-(1.4), where $f = f(t, u, u'')$ and $f = f(t, u, u')$, respectively. In [12] a complicated example was designed to satisfy the sufficient conditions of the existence.

Remark that except for boundary conditions (1.2), other types of integral boundary conditions for the fully or not fully third-order differential equations have also attracted attention from many authors. Among them there are the boundary conditions

$$u(0) = 0, u'(0) = 0, u(1) = \int_0^1 g(s)u(s)ds. \tag{1.5}$$

Some authors, e.g., Guendouz et al. [10] studied the existence of positive solutions of the BVPs (1.1), (1.5); Smirnov [17] investigated the existence and uniqueness of solutions by using the Green function of the differential equation with nonlocal boundary conditions. A year after, Smirnov studied the existence of multiple positive solutions of the equation (1.3) with the boundary conditions

$$u(0) = 0, u'(0) = 0, u(1) = \lambda[u], \tag{1.6}$$

where $\lambda[u] = \int_0^1 u(s)d\Lambda(s)ds$ is a linear functional on $C[0,1]$ given by Stieltjes integral with $\hat{\imath}$ a suitable function of bounded variation. Boundary conditions (1.6) include as special cases multipoint conditions and integral conditions. In [21] Zhang and Sun investigated the existence of monotone positive solutions for the following nonlocal problem

$$\begin{aligned}
&u''' + f(t, u, u') = 0, \quad t \in (0,1),\\
&u(0) = 0,\\
&au'(0) - bu''(0) = \alpha[u],\\
&cu'(1) + du''(1) = \beta[u],
\end{aligned}$$

where $\alpha[u] = \int_0^1 u(s)dA(s)ds, \beta[u] = \int_0^1 u(s)dB(s)ds$ are linear functionals on $C^1[0,1]$ given by Riemann-Stieltjes integrals. Very recently, Szajnowska and Zim [23] studied the existence of positive solutions to the third-order differential equation of the form

$$-u''' + m^2 u' = f(t, u, u'),$$

subject to the non-local boundary conditions

$$u(0) = 0, u'(0) = \alpha[u], u'(1) = \beta[u],$$

where m is a positive parameter and α and β are functionals (not necessarily linear) acting on the space $C^1[0,1]$.

It should be said that the mentioned above works are only concerned with the existence of solutions or positive solutions but not with the finding of solutions. To our best knowledge, only in [15] Pandey proposed the finite difference method for solving the problem

$$\begin{aligned} u'''(t) &= f(t,u(t)),\ t\in(0,1),\\ u(0)=\alpha, u''(0)=\beta, u(1) &= \int_0^1 g(s)u(s)ds. \end{aligned}$$

For the linear test problem when $f=f(t)$ the author proved that the proposed finite difference method has the accuracy at least $O(h^2)$. Recently, in [7] we established the existence and uniqueness of solutions and proposed an iterative on continuous level for finding the problem (1.1), (1.5). The methodology used in this paper is similar to the one in our previous work [5], where for the fourth-order nonlinear equation with integral boundary conditions we constructed an iterative method on continuous level and its discrete version with the accuracy $O(h^2)$.

Motivated by the above facts, in this paper we construct a higher order numerical method, namely, method of fourth order of convergence for the problem (1.1)-(1.2). Before doing this we establish the existence and uniqueness of solutions of it. Many numerical examples confirm the accuracy of $O(h^4)$ for the proposed method.

2. Existence and uniqueness of solution

To study the problem (1.1)-(1.2) we reduce it to an operator equation. For this purpose we set

$$\varphi(t) = f(t,u(t),u'(t),u''(t)), \tag{2.1}$$

and

$$\alpha = \int_0^1 g(t)u(t)dt. \tag{2.2}$$

Then the problem becomes

$$\begin{aligned} &u'''(t) = \varphi(t), \quad 0<t<1,\\ &u(0)=c_1, u''(0)=c_2,\ u(1)=\alpha+c_3. \end{aligned} \tag{2.3}$$

Now, we introduce the mixed space of functions $\varphi(t)\in C[0,1]$ and numbers $\alpha\in\mathbb{R}$ denoted by $\mathcal{B}=C[0,1]\times\mathbb{R}$. For elements $w=(\varphi,\alpha)^T\in\mathcal{B}$ we define the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\alpha|), \tag{2.4}$$

where $\|\varphi\| = \max_{0\le t\le 1}|\varphi(t)|$, r is a number, $r>0$, to be determined later in every particular case.

Now, let $w=(\varphi,\alpha)^T$ be an arbitrary element of $\mathcal{B}$. In the space $\mathcal{B}$ define the operator $A:\varphi\ \to A\varphi$ by the formula

$$Aw = \begin{pmatrix} f(t,u(t),u'(t),u''(t)) \\ \int_0^1 g(t)u(t)dt \end{pmatrix}, \tag{2.5}$$

where $u(t)$ is the solution of the problem of the form (2.3).

Lemma 2.1. *If $w = (\varphi, \alpha)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, i.e., is a solution of the operator equation*

$$Aw = w \tag{2.6}$$

in $\mathcal{B}$, then the function $u(t)$ defined from the problem (2.3) *is a solution of the original problem* (1.1)-(1.2).

Conversely, if $u(t)$ is a solution of (1.1)-(1.2)*, then the pair $(\varphi, \alpha)^T$ given by* (2.1) *and* (2.2) *is a solution of the operator equation* (2.6).

Proof. Suppose $w = (\varphi, \alpha)^T$ is a fixed point of the operator A, i.e., $Aw = w$. Then, from the definition of the operator A and by (2.5) and (2.3) it follows that $u(t)$ is the solution of the problem (1.1)-(1.2).

Conversely, suppose that $u(t)$ is a solution of (1.1)-(1.2). Then, by settings (2.1) and (2.2) and the definition of A by (2.5) we have $Aw = w$, where $w = (\varphi, \alpha)^T$. The lemma is proved. □

Hence, according to the above lemma, the solution of the problem (1.1)-(1.2) is reduced to finding the fixed point of the operator A.

Let $G_0(t, s)$ be the Green function of the operator $u'''(t) = 0$ associated with the homogeneous boundary conditions $u(0) = u''(0) = u(1) = 0$. It has the form

$$G_0(t,s) = \begin{cases} \frac{1}{2}(1-t)(s^2-t), & 0 \le s \le t \le 1, \\ -\frac{1}{2}t(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Then the solution of the problem (2.3) is represented in the form

$$u(t) = \int_0^1 G_0(t,s)\varphi(s)ds + P_2(t) + \alpha t, \quad 0 \le t \le 1, \tag{2.8}$$

where $P_2(t)$ is the polynomial of second degree satisfying the conditions

$$P_2(0) = c_1, P_2''(0) = c_2, P_2(1) = c_3.$$

It is easy to see that

$$P_2(t) = \frac{1}{2}c_2 t^2 + (c_3 - c_1 - \frac{1}{2}c_2)t + c_1. \tag{2.9}$$

Taking derivatives of (2.8) we obtain

$$\begin{aligned} u'(t) &= \int_0^1 G_1(t,s)\varphi(s)ds + P_2'(t) + \alpha, \\ u''(t) &= \int_0^1 G_2(t,s)\varphi(s)ds + P_2''(t), \end{aligned} \tag{2.10}$$

where $G_i(t, s)$ are the first and second derivatives of $G_0(t, s)$ with respect to t, $(i = 1, 2)$. They have the forms

$$G_1(t,s) = \begin{cases} -\frac{1}{2}(1-2t+s^2), & 0 \le s \le t \le 1, \\ -\frac{1}{2}(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{2.11}$$

$$G_2(t,s)=\begin{cases}1, & 0\le s<t\le 1,\\ 0, & 0\le t<s\le 1.\end{cases} \tag{2.12}$$

From (2.8) and (2.10) we have the estimates

$$\begin{aligned}\|u\| &\le M_0\|\varphi\|+\|P_2\|+|\alpha|,\\ \|u'\| &\le M_1\|\varphi\|+\|P_2'\|+|\alpha|,\\ \|u''\| &\le M_2\|\varphi\|+\|P_2''\|\end{aligned} \tag{2.13}$$

where

$$M_i=\max_{0\le t\le 1}\int_0^1|G_i(t,s)|ds, \quad (i=0,1,2)$$

and $\|.\|$ is the max-norm in $C[0,1]$. It is easy to verify that

$$M_0=\frac{1}{9\sqrt{3}},\ M_1=\frac{1}{3}, M_2=1. \tag{2.14}$$

Now, as usual, denote by $B[0,M]$ the closed ball centered at 0 with radius M in $\mathcal{B}$, i.e.,

$$B[0,M]=\big\{w=(\varphi,\alpha)^T|\ \|\varphi\|\le M, r|\alpha|\le M\big\}.$$

In the space $[0,1]\times\mathbb{R}^3$, for any number $M>0$, define the domain

$$\begin{aligned}\mathcal{D}_M=\Big\{&(t,u,y,z)\mid 0\le t\le 1,\ |u|\le (M_0+\frac{1}{r})M+\|P_2\|,\\ &|y|\le (M_1+\frac{1}{r})M+\|P_2'\|,\ |z|\le M_2M+\|P_2''\|\Big\}.\end{aligned} \tag{2.15}$$

Further, set

$$a_0=\int_0^1|g(t)|dt,\ a_1=\int_0^1 t|g(t)|dt,\ a_2=\int_0^1|g(t)P_2(t)|dt \tag{2.16}$$

and suppose that the following hypothesis is satisfied:

(H0) $ra_0M_0+a_1\le\frac{2}{3},\ ra_2\le\frac{1}{3}, q_1=ra_0M_0+a_0<1.$

Theorem 2.1 (Existence and uniqueness)**.** *In addition to hypothesis* **(H0)** *suppose that there exist numbers* $M\ge 1, L_0, L_1, L_2\ge 0$ *such that the following hypotheses are satisfied:*

(H1) $|f(t,x,y,z)|\le M,\ \forall(t,x,y,z)\in\mathcal{D}_M.$

(H2) $|f(t,x_2,y_2,z_2)-f(t,x_1,y_1,z_1)|\le L_0|x_2-x_1|+L_1|y_2-y_1|+L_2|z_2-z_1|,$ $\forall(t,x_i,y_i,z_i)\in\mathcal{D}_M,\ i=1,2.$

(H3) $q:=\max\{q_1,q_2\}<1$ *where* q_1 *was defined in* **(H0)** *and*

$$q_2=L_0(M_0+\frac{1}{r})+L_1(M_1+\frac{1}{r})+L_2M_2. \tag{2.17}$$

Then the problem (1.1)-(1.2) *has a unique solution* $u\in C^3[0,1]$.

Proof. In order to prove the theorem we shall prove that the operator A defined by (2.5) has a unique fixed point $w = (\varphi, \alpha)^T$ in the space $\mathcal{B}$, then according to Lemma 2.1 the solution of the problem (2.3) is the solution of the problem (1.1)-(1.2).

First we prove that the operator A maps the closed ball $B[0, M] \subset \mathcal{B}$ into itself. Let $w = (\varphi, \alpha)^T \in B[0, M]$, i.e., $\|w\|_{\mathcal{B}} \le M$, or equivalently, $\|\varphi\| \le M, r|\alpha| \le M$. Let $u(t)$ be the solution of the problem (2.3). Then, as shown above, the estimates (2.13) hold. In these estimates, by replacing $\|\varphi\| \le M, |\alpha| \le \frac{1}{r}M$ we obtain

$$\begin{aligned}\|u\| &\le (M_0 + \frac{1}{r})M + \|P_2\|,\\ \|u'\| &\le M_1\|\varphi\| + \|P_2'\| + |\alpha|,\\ \|u''\| &\le M_2 M + \|P_2''\|.\end{aligned}$$

Therefore, for any $t \in [0, 1]$ we have $(t, u(t), u'(t), u''(t)) \in \mathcal{D}_M$. Now we estimate $\|Aw\|_{\mathcal{B}}$. By definition we have

$$\|Aw\|_{\mathcal{B}} = \max\Big(\|f(., u(.), u'(.), u''(.))\|, r\Big|\int_0^1 g(t)u(t)dt\Big|\Big).$$

Since $(t, u(t), u'(t), u''(t)) \in \mathcal{D}_M, \forall t \in [0, 1]$ by Hypothesis (H1) we have

$$|(t, u(t), u'(t), u''(t))| \le M, \forall t \in [0, 1].$$

Consequently,

$$\|f(., u(.), u'(.), u''(.))\| \le M. \tag{2.18}$$

Next we estimate

$$I := r|\int_0^1 g(t)u(t)dt|.$$

Substituting the expression of $u(t)$ by (2.8) we have the estimate

$$\begin{aligned}I \le& r\int_0^1 \Big|\int_0^1 G_0(t,s)\varphi(s)ds||g(t)\Big|dt + r\int_0^1 |g(t)P_2(t)|dt\\ &+ r|\alpha|\int_0^1 |g(t)|dt.\end{aligned}$$

Taking into account the notations (2.16) we obtain

$$I \le (ra_0M_0 + a_1)M + ra_2 \le (ra_0M_0 + a_1 + ra_2)M \le M \tag{2.19}$$

in view of the assumption $M \ge 1$ and hypothesis (H0).

From (2.18) and (2.19) it follows that $\|Aw\|_{\mathcal{B}} \le M$. It means that the operator A maps $B[0, M]$ into itself.

Second, we show that A is a contraction mapping in $B[0, M]$.

Let $w_i = (\varphi_i, \alpha_i)^T \in B[0, M]$, $(i = 1, 2)$. We have $w_2 - w_1 = (\varphi_2 - \varphi_1, \alpha_2 - \alpha_1)^T$. From the definition of the norm in the space $\mathcal{B}$ it follows

$$\|\varphi_2 - \varphi_1\| \le \|w_2 - w_1\|_{\mathcal{B}},\ |\alpha_2 - \alpha_1| \le \frac{1}{r}\|w_2 - w_1\|_{\mathcal{B}}. \tag{2.20}$$

Further, let $u_i(t)$ be the solutions of the problems

$$\begin{aligned}&u_i'''(t) = \varphi_i(t), \quad 0 < t < 1,\\ &u_i(0) = c_1, u_i''(0) = c_2,\ u_i(1) = \alpha_i + c_3.\end{aligned}$$

Then, as (2.8), we have for $i = 1, 2$

$$u_i(t) = \int_0^1 G_0(t,s)\varphi_i(s)ds + P_2(t) + \alpha_i t, \quad 0 \le t \le 1. \tag{2.21}$$

Hence,

$$u_2(t) - u_1(t) = \int_0^1 G_0(t,s)(\varphi_2(s) - \varphi_1(s))ds + (\alpha_2 - \alpha_1)t.$$

Consequently,

$$\|u_2 - u_1\| \le M_0\|\varphi_2 - \varphi_1\| + |\alpha_2 - \alpha_1|.$$

Taking into account (2.20) we obtain

$$\|u_2 - u_1\| \le (M_0 + \frac{1}{r})\|w_2 - w_1\|_{\mathcal{B}}. \tag{2.22}$$

Analogously, we have

$$\|u_2' - u_1'\| \le (M_1 + \frac{1}{r})\|w_2 - w_1\|_{\mathcal{B}},\ \|u_2'' - u_1''\| \le M_2\|w_2 - w_1\|_{\mathcal{B}}. \tag{2.23}$$

Now we are ready to estimate $\|Aw_2 - Aw_1\|_{\mathcal{B}}$. We have

$$Aw_2 - Aw_1 = \begin{pmatrix} f(t, u_2(t), u_2'(t), u_2''(t)) - f(t, u_1(t), u_1'(t), u_1''(t)) \\ \int_0^1 g(t)(u_2(t) - u_1(t))dt \end{pmatrix}. \tag{2.24}$$

As shown above $(t, u_i(t), u_i'(t), u_i''(t)) \in \mathcal{D}_M\ \forall t \in [0,1]$, using Hypothesis (H2) and the estimates (2.22)-(2.23) we obtain

$$\begin{aligned} T_1 :=& |f(t, u_2(t), u_2'(t), u_2''(t)) - f(t, u_1(t), u_1'(t), u_1''(t))| \\ \le& (L_0(M_0 + \frac{1}{r}) + L_1(M_1 + \frac{1}{r}) + L_2M_2)\|w_2 - w_1\|_{\mathcal{B}} \\ =& q_2\|w_2 - w_1\|_{\mathcal{B}} \text{ (see notation (2.17))} . \end{aligned} \tag{2.25}$$

It remains to estimate the second component in (2.24). In view of (2.22) and the notation a_0 in (2.16) we have

$$T_2 := \Big|\int_0^1 g(t)(u_2(t) - u_1(t))dt\Big| \le a_0(M_0 + \frac{1}{r})\|w_2 - w_1\|_{\mathcal{B}}. \tag{2.26}$$

From (2.25), (2.26), the notation q_1 in Hypothesis (H0) and Hypothesis (H3) it follows that

$$\|Aw_2 - Aw_1\|_{\mathcal{B}} \le q\|w_2 - w_1\|_{\mathcal{B}} \tag{2.27}$$

with $q < 1$. So, the operator A is a contraction mapping in $B[0, M]$. By the contraction mapping principle, the operator has a unique fixed point. Thus, the theorem is proved. □

3. Iterative method on continuous level

To solve the problem (1.1)-(1.2) we propose the following iterative method on continuous level:

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0, M]$, for example,
$$\varphi_0(t) = f(t, 0, 0, 0),\ \ \alpha_0 = 0. \tag{3.1}$$
2. Knowing $\varphi_k(t)$ and $\alpha_k(t)$ $(k = 0, 1, ...)$ compute
$$u_k(t) = \int_0^1 G_0(t, s)\varphi_k(s)ds + P_2(t) + \alpha_k t, \tag{3.2}$$
$$y_k(t) = \int_0^1 G_1(t, s)\varphi_k(s)ds + P_2'(t) + \alpha_k, \tag{3.3}$$
$$z_k(t) = \int_0^1 G_2(t, s)\varphi_k(s)ds + P_2''(t). \tag{3.4}$$
3. Update
$$\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t)), \tag{3.5}$$
$$\alpha_{k+1} = \int_0^1 g(t)u_k(t)dt. \tag{3.6}$$

It is easy to recognize that the above iterative method is a realization of the successive approximation method for finding the fixed point $w = (\varphi, \alpha)^T$ of the operator A. Therefore, it converges and there holds the estimate
$$\|w_k - w\|_{\mathcal{B}} \le p_k,$$
where $w_k - w = (\varphi_k - \varphi, \alpha_k - \alpha)^T$, and
$$p_k = \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}}. \tag{3.7}$$
From here and the definition of the norm in $\mathcal{B}$ it follows
$$\|\varphi_k - \varphi\| \le \|w_k - w\|_{\mathcal{B}} \le p_k,$$
$$|\alpha_k - \alpha| \le \frac{1}{r}\|w_k - w\|_{\mathcal{B}} \le \frac{1}{r}p_k.$$
Now, taking into account the representations (2.8), (2.10) and the formulas (3.2)-(3.4) we obtain the estimates
$$\begin{aligned} \|u_k - u\| &\le \left(M_0 + \frac{1}{r}\right)p_k, \\ \|y_k - u'\| &\le \left(M_1 + \frac{1}{r}\right)p_k, \\ \|z_k - u''\| &\le M_2 p_k. \end{aligned} \tag{3.8}$$
Thus, we have proved the theorem:

Theorem 3.1. *Under the hypotheses of Theorem 2.1 the iterative method* (3.1)-(3.6) *converges, and for the approximate solution* $u_k(t)$ *and its derivatives* $u_k' = y_k, u_k'' = z_k$ *there hold the estimates* (3.8).

4. Construction of discrete iterative method

4.1. Trapezoidal quadrature formula with corrections

Let $h = 1/n$, where n is a positive integer, and $s_i = (i-1)h, i = 1, \dots, n+1$. Then the Euler-Maclaurin formula has the form (see [13])

$$\int_0^1 \Phi(s)ds = T_\Phi(h) - \sum_{l=1}^{p-1} \frac{B_{2l}}{(2l!)} \Big(\Phi^{(2l-1)}(a) - \Phi^{(2l-1)}(0) \Big) + O(h^{2p}), \tag{4.1}$$

where $\Phi \in C^{2p}[0,1]$, B_{2l} are Bernoulli numbers,

$$T_\Phi(h) = \frac{h}{2}(\Phi_1 + \Phi_{n+1}) + \sum_{i=2}^{n} h\Phi_i, \tag{4.2}$$

with $\Phi_i = \Phi(s_i)$. Now, for a fixed $t \in [0,1]$, let

$$\Phi(s) = g(t,s)\varphi(s),$$

where $g(t,s)$ is continuous in $[0,1]\times[0,1]$, and may have discontinuous derivatives at the point $s = t$, meanwhile $\varphi(s) \in C^{2p}[0,1]$. Then using the above Euler-Maclaurin formula, Sidi and Pennline [16] obtained the following formula

$$\begin{aligned} \int_0^1 \Phi(s)ds =& T_\Phi(h) - \sum_{l=1}^{p-1} \frac{B_{2l}}{(2l!)} \Big\{ [\Phi^{(2l-1)}(1) - \Phi^{(2l-1)}(0)] \\ & - [\Phi^{(2l-1)}(t^+) - \Phi^{(2l-1)}(t^-)] \Big\} + O(h^{2p}), \end{aligned} \tag{4.3}$$

where $\varphi(t^+)$ and $\varphi(t^-)$ are the one-sided limits of the function $\varphi(t)$ at t. In the particular case $p = 2$ we have

$$\int_0^1 \Phi(s)ds = T_\Phi(h) - \frac{h^2}{12} \Big\{ [\Phi'(1) - \Phi'(0)] - [\Phi'(t^+) - \Phi'(t^-)] \Big\} + O(h^4). \tag{4.4}$$

If the function $\Phi(s)$ has a jump at point $t \in (0,a)$ then in the above formula instead of $T_\Phi(h)$ it should be $T_{\Phi^*}(h)$, where

$$\Phi^*(s) = \begin{cases} \Phi(s), & s \neq t, \\ \frac{1}{2}[\Phi(t^+) + \Phi(t^-)], & s = t. \end{cases}$$

4.2. Construction of some quadrature formulas

4.2.1. Computing the integrals of the type in (3.2)

Now we apply the formulas (4.4) to construct numerical methods of order 4. Recall that the Green function associated with the problem under consideration has the form

$$G_0(t,s) = \begin{cases} \frac{1}{2}(1-t)(s^2 - t), & 0 \le s \le t \le 1, \\ -\frac{1}{2}t(1-s)^2, & 0 \le t \le s \le 1. \end{cases}$$

We have $G_0(t,1)=0$ and

$$\frac{\partial G_0(t,s)}{\partial s}=\begin{cases} s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1.\end{cases}$$

For a fixed t put

$$\Phi(s)=G_0(t,s)\varphi(s),$$

where $\varphi\in C^1[0,1]$. It is easy to verify that

$$\begin{aligned}&\Phi'(1)-\Phi'(0)=-\frac{1}{2}t(1-t)\varphi'(0),\\ &\Phi'(t^+)-\Phi'(t^-)=0.\end{aligned}$$

Therefore, from (4.4) we obtain

$$\int_0^1 G_0(t,s)\varphi(s)ds=T_\Phi(h)-\frac{h^2}{24}t(1-t)\varphi'(0)+O(h^4). \tag{4.5}$$

We have immediately

$$\int_0^1 G_0(t_i,s)\varphi(s)ds=0,\ (i=1;n+1) \tag{4.6}$$

since $G(0,s)=G(1,s)=0$. At the points $t_i=(i-1)h, i=2,...,n$, we obtain

$$\int_0^1 G_0(t_i,s)\varphi(s)ds=L_4(G,t_i)\varphi+O(h^4), \tag{4.7}$$

where

$$L_4(G,t_i)\varphi=\sum_{j=1}^{n+1}h\rho_jG_0(t_i,t_j)\varphi_j-\frac{h^2}{24}\Big\{t_i(1-t_i)D_2^{(1)}\varphi_1\Big\},\ i=2,...,n. \tag{4.8}$$

Here, $D_2^{(1)}\varphi_i\,(i=1,...,n+1)$ is the difference approximation with the accuracy $O(h^2)$ of the first derivative of the function $\varphi(t)$ at the point t_i, i.e., $D_2^{(1)}\varphi_i=\varphi'(t_i)+O(h^2)$. In the above formula ρ_j are the weights

$$\rho_j=\begin{cases}1/2,\ j=0,n\\ 1,\ j=1,2,...,n-1\end{cases}$$

and for short we denote $\varphi_j=\varphi(t_j), j=1,...,n+1$.

Due to (4.6) we set

$$L_4(G,t_1)\varphi=L_4(G,t_{n+1})\varphi=0. \tag{4.9}$$

4.2.2. Computing the integrals of the type in (3.3)

Now we establish the formula for computing $\int_0^1 G_1(t,s)\varphi(s)ds$.

Recall that

$$G_1(t,s) = \begin{cases} -\dfrac{1}{2}(1-2t+s^2), & 0 \le s \le t \le 1, \\ -\dfrac{1}{2}(1-s)^2, & 0 \le t \le s \le 1. \end{cases}$$

Therefore,

$$\frac{\partial G_1(t,s)}{\partial s} = \begin{cases} -s, & 0 \le s < t \le 1, \\ 1-s, & 0 \le t < s \le 1. \end{cases}$$

And it is easy to obtain

$$\int_0^1 G_1(t,s)\varphi(s)ds = T_{\Phi_1}(h) - \frac{h^2}{12}\Big\{\frac{1}{2}(1-2t)\varphi'(0) - \varphi(t)\Big\} + O(h^4), \tag{4.10}$$

where $\Phi_1(s) = G_1(t,s)\varphi(s)$, t is fixed. At the points $t_i, i = 1, ..., n+1$ we obtain

$$\int_0^1 G_1(t_i,s)\varphi(s)ds = L_4(G_1,t_i)\varphi + O(h^4), \tag{4.11}$$

where

$$L_4(G_1,t_i)\varphi = \sum_{j=1}^{n+1} h\rho_j G_1(t_i,t_j)\varphi_j - \frac{h^2}{12}\Big\{\frac{1}{2}(1-2t_i)D_2^{(1)}\varphi_1 - \varphi_i\Big\},\ i = 1, ..., n+1. \tag{4.12}$$

4.2.3. Computing the integrals of the type in (3.4)

Now we establish the formula for computing $\int_0^1 G_2(t,s)\varphi(s)ds$.

Recall that

$$G_2(t,s) = \begin{cases} 1, & 0 \le s < t \le 1, \\ 0, & 0 \le t < s \le 1. \end{cases}$$

Therefore,

$$\frac{\partial G_2(t,s)}{\partial s} = 0,\ s \ne t.$$

Analogously as for the integrals of the types in (3.2) and (3.3) we obtain

$$\int_0^1 G_2(t_i,s)\varphi(s)ds = L_4(G_2,t_i)\varphi + O(h^4), \tag{4.13}$$

where

$$L_4(G_2,t_i)\varphi = \sum_{j=1}^{n+1} h\rho_j G_2^*(t_i,t_j)\varphi_j - \frac{h^2}{12}\Big\{D_2^{(1)}\varphi_i - D_2^{(1)}\varphi_1\Big\},\ i = 2, ..., n+1. \tag{4.14}$$

Here,

$$G_2^*(t,s) = \begin{cases} 1, & 0 \le s < t \le 1, \\ \frac{1}{2}, & s = t, \\ 0, & 0 \le t < s \le 1. \end{cases}$$

We also set

$$L_4(G_2,t_1)\varphi = 0 \tag{4.15}$$

since $G_2(t_1,s)=0$.

4.2.4. Computing the integrals of the type in (3.6)

Now we establish the formula for computing the integral

$$\int_0^1 u(s)ds \tag{4.16}$$

with the accuracy $O(h^4)$. Setting $\Phi(s)=g(s)u(s)$ we have

$$\int_0^1 \Phi(s)ds = T_\Phi(h) - \frac{h^2}{12}\big\{g'(1)u(1)+g(1)u'(1)-g'(0)u(0)-g(0)u'(0)\big\} + O(h^4). \tag{4.17}$$

Approximating $u'(1)$ and $u'(0)$ with the accuracy $O(h^2)$ we have

$$\begin{aligned}\int_0^1 \Phi(s)ds =& T_\Phi(h) - \frac{h^2}{12}\big\{g'(1)u(1)+g(1)D_2^{(1)}u_{n+1} \\ &- g'(0)u(0)-g(0)D_2^{(1)}u_1\big\} + O(h^4).\end{aligned} \tag{4.18}$$

Rewrite the above formula in the form

$$\int_0^1 g(s)u(s)ds = L_4(g)u + O(h^4), \tag{4.19}$$

where

$$\begin{aligned}L_4(g)u =& \sum_{i=1}^{n+1} h\rho_j g(t_j)u(t_j) - \frac{h^2}{12}\Big\{g'(1)u(t_{n+1})+g(1)D_2^{(1)}u_{n+1} \\ &- g'(0)u(t_1)-g(0)D_2^{(1)}u_1\Big\}.\end{aligned} \tag{4.20}$$

4.3. Discrete iterative method for the BVP (1.1)-(1.2)

Cover the interval $[0,1]$ by the uniform grid $\bar{\omega}_h = \{t_i = (i-1)h,\ h=1/n, i = 1,2,...,n+1\}$ and denote by $\Phi_k(t), U_k(t), Y_k(t), Z_k(t)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_k(t),\ u_k(t), y_k(t), z_k(t)$ on this grid, respectively. We denote also by $\hat{\alpha}_k$ the approximation of α_k. Consider the following

Iterative Method:

1. Given
$$\Phi_0(t_i) = f(t_i,0,0,0),\ i=1,...,n+1;\ \hat{\alpha}_0 = 0. \tag{4.21}$$
2. Knowing $\Phi_k(t_i),\ \hat{\alpha}_k$ $(k=0,1,...;\ i=1,...,n+1)$, compute approximately the definite integrals (3.2)-(3.4) by the trapezoidal formulas with corrections:
$$\begin{aligned}U_k(t_i) &= L_4(G,t_i)\Phi_k + P_2(t_i) + \hat{\alpha}_k t_i,\\ Y_k(t_i) &= L_4(G_1,t_i)\Phi_k + P_2'(t_i) + \hat{\alpha}_k,\\ Z_k(t_i) &= L_4(G_2,t_i)\Phi_k + P_2''(t_i),\ i=1,...,n+1,\end{aligned} \tag{4.22}$$

where $L_4(G,t_i)\Phi_k, L_4(G_1,t_i)\Phi_k, L_4(G_2,t_i)\Phi_k$ are defined by (4.8)-(4.9), (4.12) and (4.14)-(4.15), respectively, by replacing the function φ on the grid by the grid function Φ_k.

3. Update

$$\begin{aligned} \Phi_{k+1}(t_i) &= f(t_i, U_k(t_i), Y_k(t_i), Z_k(t_i)),\ i = 1, ..., n+1. \\ \hat{\alpha}_{k+1} &= L_4(g)U_k. \end{aligned} \tag{4.23}$$

Now we study the convergence of the above iterative method.

Proposition 4.1. *Under the assumptions that the function $f(t,u,y,z)$ has all continuous partial derivatives up to fourth order in the domain $\mathcal{D}_M$ and the function $g(s) \in C^4[0,1]$, for any $k = 0,1,...$ there hold the estimates*

$$\|\Phi_k - \varphi_k\| = O(h^4),\ |\hat{\alpha}_k - \alpha| = O(h^4), \tag{4.24}$$

$$\begin{aligned} &\|U_k - u_k\| = O(h^4),\ \|Y_k - y_k\| = O(h^4), \\ &\|Z_k - z_k\| = O(h^4), \end{aligned} \tag{4.25}$$

where $\|.\| = \|.\|_{C(\bar{\omega}_h)}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof. The proposition can be proved by induction in a similar way as Proposition 3 in [5] if taking into account the order 4 of quadrature formulas used in design of the discrete method and the linearity of $L_4(G,t_i)$ as an operator acting on the grid function φ_k.

Now combining the above proposition and Theorem 2.1 we obtain the following theorem. □

Theorem 4.1. *Under the assumptions of the above proposition and Theorem 2.1, for the approximate solution of the problem (1.1)-(1.2) obtained by the above discrete iterative method on the uniform grid with gridsize h, we have the estimates*

$$\begin{aligned} &\|U_k - u\| \le \left(M_0 + \frac{1}{r}\right) p_k + O(h^4),\ \|Y_k - u'\| \le \left(M_1 + \frac{1}{r}\right) p_k + O(h^4), \\ &\|Z_k - u''\| \le M_2 p_k + O(h^4), \end{aligned} \tag{4.26}$$

where M_i $(i = 0,1,2)$ are defined by (2.14) and $p_k d$ are defined by (3.7).

Proof. The first above estimate is easily obtained if representing

$$U_k(t_i) - u(t_i) = (u_k(t_i) - u(t_i)) + (U_k(t_i) - u_k(t_i))$$

and using the first estimate in Theorem 2.1 and the first estimate in (4.25). The remaining estimates are obtained in the same way. Thus, the theorem is proved. □

5. Examples

First we show an example for illustrating the applicability of the theoretical results on existence and uniqueness of solution, and the fourth-order convergence of the proposed numerical method. After that we show some more examples to confirm the fourth convergence order of the method. Among these examples there are also two examples from [15] for comparing the convergence order. In all the numerical experiments, the iterative method is performed until $\|\Phi_{k+1} - \Phi_k\| \le 10^{-14}$.

Example 5.1. Consider the fully third-order nonlinear boundary value problem

$$\begin{aligned} u'''(t) =& -\frac{2}{3}\cos(t) - \frac{1}{2}\cos(\sin(t)) - \frac{1}{6}e^{-|\sin(t)|} \\ &+ \frac{1}{2}\cos(u) - \frac{1}{3}u' + \frac{1}{6}e^{-|u''|},\ t \in [0,1], \\ u(0) =& 0, u''(0) = 0,\ u(1) = \int_0^1 t^2 u(t)dt + 2 - \cos(1) - sin(1), \end{aligned} \tag{5.1}$$

In this problem

$$\begin{aligned} f(t,u,y,z) =& -\frac{2}{3}\cos(t) - \frac{1}{2}\cos(\sin(t)) - \frac{1}{6}e^{-|\sin(t)|} + \frac{1}{2}\cos(u) - \frac{1}{3}y + \frac{1}{6}e^{-|z|}, \\ c_1 =& c_2 = 0, c_3 = 2 - \cos(1) - \sin(1), \\ g(t) =& t^2. \end{aligned}$$

It is easy to see that the function $u(t) = \sin(t)$ is an exact solution of the problem. We shall verify that all the conditions of Theorem 2.1 are satisfied. Indeed, for the problem

$$\begin{aligned} P_2(t) =& (2 - \cos(1) - \sin(1))t = 0.6182t, \\ \|P_2\| =& \|P_2'\| = 0.6182, \|P_2''\| = 0, \\ a_0 =& \frac{1}{3}, a_1 = \frac{1}{4}, a_2 = 0.1546. \end{aligned}$$

Hence, for $r = 2$, Hypothesis (H0) in Theorem 2.1 is satisfied.

The domain $\mathcal{D}_M$ is defined by

$$\begin{aligned} \mathcal{D}_M = \Big\{ & (t,u,y,z) \mid 0 \le t \le 1,\ |u| \le (M_0 + \frac{1}{r})M + 0.6182, \\ & |y| \le (M_1 + \frac{1}{r})M + 0.6182,\ |z| \le M_2 M \Big\}. \end{aligned}$$

In $\mathcal{D}_M$ we have the estimate

$$|f(t,u,y,z)| \le 2.2061 + \frac{1}{3}(M_1 + \frac{1}{r})M.$$

Taking $M = 3.1$ we have $|f(t,u,y,z)| < M$. So, Hypothesis (H1) in Theorem 2.1 is satisfied. Next, it is possible to verify that the function $f(t,u,y,z)$ satisfies the Lipshitz conditions with the coefficients $L_0 = 1/2, L_1 = 1/3$ and $L_2 = 1/6$. We have also $q_2 < 1$. Thus, Hypotheses (H2) and (H3) are satisfied. As a result, all the hypotheses of Theorem 2.1 are satisfied. Therefore, the problem has a unique solution. This solution, of course, is the function $u(t) = \sin(t)$.

The results of convergence of the discrete iterative method are given in Table 1, where $n+1$ is the number of grid points, K is the number of iterations performed, $Error = \|U_K - u\|$ and $Order$ is the order of convergence calculated by the formula

$$Order = \log_2 \frac{\|U_K^{n/2} - u\|}{\|U_K^n - u\|}.$$

Table 1. The convergence in Example 1.

n	K	Error	h^4	Order
8	29	1.8590e-06	2.4414e-04	
16	29	9.4341e-08	1.5259e-05	4.3005
32	29	5.1883e-09	9.5367e-07	4.1846
64	29	3.0177e-10	5.9605e-08	4.1038
128	29	1.8152e-11	3.7253e-09	4.0552
256	29	1.1131e-12	2.3283e-10	4.0275
512	29	6.9611e-14	1.4552e-11	3.9991
1024	29	5.1070e-15	9.0949e-13	3.7688

The superscripts $n/2$ and n of U_K mean that U_K is computed on the grid with the corresponding number of grid points. From Table 1 it is obvious that the iterative method has the accuracy $O(h^4)$ and the convergence order is 4.

Example 5.2. Consider the fully third-order nonlinear boundary value problem

$$u'''(t) = -\frac{4}{3}e^{2t} + e^t + u^2 + u - u' + \frac{1}{3}[u'']^2, t \in [0,1]$$
$$u(0) = 1, u''(0) = 1,\ u(1) = \int_0^1 g(s)u(s)ds + 2,$$

where $g(s) = s^2$. The problem has the exact solution $u = e^t$. The results of convergence of the discrete iterative method are given in Table 2.

Table 2. The convergence in Example 2

n	K	Error	h^4	Order
n	iter	error	h4	order
8	28	9.3394e-06	2.4414e-04	
16	28	6.5803e-07	1.5259e-05	3.8271
32	28	4.3618e-08	9.5367e-07	3.9152
64	28	2.8067e-09	5.9605e-08	3.9580
128	28	1.7798e-10	3.7253e-09	3.9791
256	28	1.1204e-11	2.3283e-10	3.9896
512	28	7.0211e-13	1.4552e-11	3.9962
1024	28	4.3521e-14	9.0949e-13	4.0119

Example 5.3. (Example 2 in [15]) Consider the third-order linear boundary value problem

$$u'''(t) = u(t) - 3e^t, t \in [0, 1],$$
$$u(0) = 1, u''(0) = -1, \ u(1) = \int_0^1 g(s)u(s)ds$$

with the exact solution $u(t) = (1-t)e^t$.

Notice that for this exact solution $u(1) = 0$, and for the function $g(s) = \frac{e^{s^2}-1}{2}$ we have $\int_0^1 g(s)u(s)ds = 0.0985$. Therefore, the integral boundary condition must be $u(1) = \int_0^1 g(s)u(s)ds - 0.0985$. With this corrected integral boundary condition we carry out the computations by the proposed iterative method. The results of the numerical experiments are reported in Table 3. In order to compare the convergence

Table 3. The convergence in Example 3

n	K	Error	h^4	Order
8	19	7.5413e-06	2.4414e-04	
16	19	5.6339e-07	1.5259e-05	3.7426
32	19	3.8978e-08	9.5367e-07	3.8534
64	19	2.5691e-09	5.9605e-08	3.9233
128	19	1.6447e-10	3.7253e-09	3.9654
256	19	9.8833e-12	2.3283e-10	4.0566
512	19	2.0707e-13	1.4552e-11	5.5768
1024	19	7.5800e-13	9.0949e-13	-1.8721

of the proposed method with that of the Pandey's method in [15] we make Table 4.

Table 4. Comparison of convergence of the proposed method with Pandey's method for Example 3

n	Error (Prop)	Order (Prop)	Error(Pandey)	Order(Pandey)
64	2.5691e-09	3.9233		
128	1.6446e-10	3.9654	1.2575090e-04	
256	9.8830e-12	4.0567	3.2846816-05	1.93674
512	2.0756e-13	5.5734	7.6917931e-06	2.09436
1024	7.5859e-13	-1.8698	1.8957071e-06	2.02058

Example 5.4. (Example 3 in [15]) Consider the third-order nonlinear boundary

value problem

$$
\begin{aligned}
&u'''(t) = e^{-t}u^2(t) + f(t), t \in [0,1], \\
&u(0) = 0, u''(0) = -2,\ u(1) = \int_0^1 g(s)u(s)ds,
\end{aligned}
\tag{5.2}
$$

where $f(t)$ is calculated so that the exact solution of the problem is $u(t) = (1 - t)\sin(t)$. In [15] $g(s) = \frac{e^s-1}{\cos(s-2)}$. But we think that there is a typo in the expression of the function $g(s)$ because it has a singularity at $t = 2 - \pi/2 \approx 0.4292$. Due to this reason, we correct the typo and adopt

$$g(s) = \frac{e^s - 1}{\cos(s+2)}.$$

For this corrected $g(s)$ we have

$$\int_0^1 g(s)u(s)ds = -0.128705665220749,$$

therefore, the integral boundary condition in the problem must be

$$u(1) = \int_0^1 g(s)u(s)ds + 0.128705665220749. \tag{5.3}$$

The results of computation for the problem (5.2) with the corrected integral boundary condition are presented in Table 5 and the comparison of convergence of the proposed method with Pandey's method is reported in Table 6.

Table 5. The convergence in Example 4

n	K	Error	h^4	Order
8	54	1.1728e-05	2.4414e-04	
16	54	7.6427e-07	1.5259e-05	3.9397
32	54	4.8787e-08	9.5367e-07	3.9695
64	54	3.0831e-09	5.9605e-08	3.9840
128	54	1.9377e-10	3.7253e-09	3.9920
256	54	1.2147e-11	2.3283e-10	3.9957
512	54	7.6317e-13	1.4552e-11	3.9924
1024	54	5.0793e-14	9.0949e-13	3.9093

6. Conclusion

In this paper, we have considered an integral boundary value problem for a fully third-order nonlinear differential equation. By the reduction of the problem to a fixed point problem of an operator which is implicitly defined via the solutions of

Table 6. Comparison of convergence of the proposed method with Pandey's method for Example 4

n	Error (Prop)	Order (Prop)	Error(Pandey)	Order(Pandey)
64	3.0831e-09	3.9840		
128	1.9377e-10	3.9920	1.5833229e-03	
256	1.2147e-11	3.9957	3.8277125e-05	2.04840
512	7.6317e-13	3.9924	9.3068229e-06	2.04012
1024	5.0793e-14	3.9093	2.3112516e-06	2.00961

an associated boundary value problem we have established the existence of a unique solution of the problem. More importantly, we have constructed a discrete iterative method for finding the solution and proved that the method has the convergence order 4. To this end we have used the trapezoidal rule with corrections. With the use of the rule with more precise corrections we can construct methods of sixth or higher order of accuracy. In the future, we shall do this for third-order differential equations involving other integral boundary conditions and for integral boundary value problems for higher order differential equations.

Acknowledgements

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments that have helped improve this paper.

References

[1] S. S. Almuthaybiri and C. C. Tisdell, *Sharper existence and uniqueness results for solutions to third-order boundary value problems*, Math. Model. Anal., 2020, 25(3), 409–420.

[2] Z. Bai, *Existence of solutions for some third-order boundary-value problems*, Electron. J. Differential Equations, 2008, 25, 1–6.

[3] A. Boucherif, S. M. Bouguima, N. Al-Malki and Z. Benbouziane, *Third order differential equations with integral boundary conditions*, Nonlinear Anal., 2009, 71(12), 1736–1743.

[4] A. Cabada and N. D. Dimitrov, *Third-order differential equations with three-point boundary conditions*, Open Math., 2021, 19(1), 11–31.

[5] Q. A Dang and Q. L. Dang, *Existence results and iterative method for a fully fourth-order nonlinear integral boundary value problem*, Numer. Algor., 2020, 85, 887–907.

[6] Q. A Dang and Q. L. Dang, *Simple numerical methods of second and third-order convergence for solving a fully third-order nonlinear boundary value problem*, Numer. Algor., 2020, 87(1), 1479–1499.

[7] Q. A Dang and Q. L. Dang, *Existence results and iterative method for fully third order nonlinear integral boundary value problems*, Appl. Math., 2021, 66, 657–672.

[8] J. R. Graef, J. Henderson, R. Luca and Y. Tian, *Boundary-Value Problems for Third-Order Lipschitz Ordinary Differential Equations*, Proc. Edinb. Math. Soc., 2005, 58(1), 183–197.

[9] M. Gregus, *Third Order Linear Differential Equations*, Mathematics and Its Applications, Reidel, Dordrecht, the Netherlands, 1987.

[10] C. Guendouz, F. Haddouchi and S. Benaicha, *Existence of positive solutions for a nonlinear third-order integral boundary value problem*, Ann. Acad. Rom. Sci., Math. Appl., 2018, 10, 314–328.

[11] Y. Guo, Y. Liu and Y. Liang, *Positive solutions for the third-order boundary value problems with the second derivatives*, Bound. Value Probl., 2012, 2012(34).

[12] Y. Guo and F. Yang, *Positive Solutions for Third-Order Boundary-Value Problems with the Integral Boundary Conditions and Dependence on the First-Order Derivatives*, J. Appl. Math., 2013, Article ID 721909.

[13] F. B. Hildebrand, *Introduction to numerical analysis*, 2nd edition, McGraw Hill, New York, 1974.

[14] Y. Li and Y. Li, *Positive Solutions of a Third-Order Boundary Value Problem with Full Nonlinearity*, Mediterr. J. Math., 2017, 14, Article ID 128.

[15] P. K. Pandey, *The numerical solution of third-order non-local boundary value problems in ODEs by the finite difference method*, ROMAI J., 2019, 15, 73–82.

[16] A. Sidi and J. Pennline, *Improving the accuracy of quadrature method solutions of Fredholm integral equations that arise from nonlinear two-point boundary value problems*, J. Integral Equations Appl., 1999, 11, 103–139.

[17] S. Smirnov, *Green's function and existence of solutions for a third-order boundary value problem involving integral condition*, Lith. Math. J., 2022, 62, 509–518.

[18] S. Smirnov, *Existence of multiple positive solutions for a third-order boundary value problem with nonlocal conditions*, Nonlinear Anal. Model. Control, 2023, 28(3), 597–612.

[19] X. Sun and Y. Li, *Existence of solutions for a class of fully third-order boundary value problem*, J. Jilin Univ. Sci., 2021, 59(1), 13–19.

[20] J. R. L. Webb, *Non-local second-order boundary value problems with derivative-dependent nonlinearity*, Phil. Trans. R. Soc. A, 2021, 379(2191).

[21] H. Zhang and J. Sun, *Existence and iteration of monotone positive solutions for third-order nonlocal BVPs involving integral conditions*, Electron. J. Qual. Theory Differ. Equ., 2012, 2012(18), 1–9.

[22] J. Zhao, P. Wang and W. Ge, *Existence and nonexistence of positive solutions for a class of third order BVP with integral boundary conditions in Banach spaces*, Commun. Nonlinear Sci. Numer. Simul., 2011, 16(1), 402–413.

[23] G. Szajnowska and M. Zim, *Positive solutions to a third order nonlocal boundary value problem with a parameter*, Opuscula Math., 2024, 44(2), 267–283.