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Geometric Inequality for CR-Slant Warped Product Submanifolds in Nearly Lorentzian Para-Sasakian Manifold

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Abstract This paper examines CR-slant warped product submanifolds of the form $B \times_f N_\theta$ within a nearly Lorentzian para-Sasakian manifold. Here, B represents a CR-product submanifold, N_θ is a slant submanifold and f denotes the warping function. We establish an inequality relating the squared norm of the second fundamental form to the warping function, considering two cases based on the behavior of the structure vector field. Additionally, the conditions under which equality holds are investigated.

Keywords Warped product, CR-slant submanifold, nearly Lorentzian para-Sasakian manifold

MSC(2010) 53B25, 53C15, 53C40, 53C42, 53D10.

1. Introduction

The concept of warped products is well-established in both differential geometry and physics. In 1969, Bishop and O'Neill [6] introduced warped products as a tool for studying manifolds with negative curvature, extending the idea of Riemannian product manifolds. They defined these manifolds as follows: Let (B, g_1) and (F, g_2) be two Riemannian manifolds, and let f be a differentiable function on B. For the product manifold $B \times F$, with projections $\gamma_1 : B \times F \to B$ and $\gamma_2 : B \times F \to F$, the warped product of B and F, denoted by $N = B \times_f F$, is endowed with a Riemannian structure defined as follows [6]

$$g(X_1, X_2) = g_1(\gamma_{1*}X_1, \gamma_{1*}X_2) + (f \circ \gamma_1)^2 g_2(\gamma_{2*}X_1, \gamma_{2*}X_2),$$

for any vector field $X_1, X_2 \in \Gamma(TN)$, where $*$ denotes the tangent map. A warped product manifold is considered trivial or simply a Riemannian product manifold, if the warping function f is constant. It is well known that, for a vector field X_1 on B and X_3 on F, the following holds: [6]

$$\nabla_{X_3} X_1 = \nabla_{X_1} X_3 = X_1(lnf)X_3, \tag{1.1}$$

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where ∇ is the Levi-Civita connection on N. Additionally, it is well established that B is totally geodesic and F is totally umbilical in the warped product $B \times_f F$ ([6], [10]).

In [10], Chen introduced the concept of CR-warped products in Kaehler manifolds, demonstrating results on the existence of warped products and proving general sharp inequalities for the second fundamental form in relation to the warping function. Subsequently, numerous articles have appeared addressing similar inequalities in almost Hermitian and almost contact metric manifolds ([7], [4], [20], [21], [17], [14]).

Sahin introduced the concept of CR-slant warped products, which are referred to as skew CR-warped products, in Kaehler manifolds in [22]. Following this, Chen et al. explored the pointwise CR-slant warped products in Kaehler manifolds in [11]. More recently, Alqahtani and Almudawi [1] examined CR-slant warped product submanifolds in nearly trans-Sasakian manifolds. They established an inequality for the second fundamental form in two cases, depending on the behavior of the structure vector field. Several geometers have also studied CR-slant warped product submanifolds, including works in ([23], [24] [3], [2], [25]).

On the other hand, a class of almost paracontact metric manifolds known as Lorentzian para-Sasakian manifolds was introduced by Matsumoto [15]. Subsequently, Mihai et al. [16] independently introduced the same concept and derived several results on these manifolds. Lorentzian para-Sasakian manifolds have also been studied by De et al. [12], Rahman et al. [19], and others. In [20], Rahman established an inequality for contact CR-warped product submanifolds of nearly Lorentzian para-Sasakian manifolds.

Building on previous studies, we investigate CR-slant warped product submanifold of the form $B \times_f N_\theta$ within a nearly Lorentzian para-Sasakian manifold. Here, $B = N_T \times N_\perp$ is the CR-product of invariant and anti-invairant submanifolds of $\tilde{M}$, N_θ is a slant submanifold and f represents the warping function. We derive a sharp estimate for the squared norm of the second fundamental form in relation to the warping function, considering two cases based on whether the structure vector field is tangent to the invariant or anti-invariant submanifold. We also investigate the equality cases of these inequalities.

2. Preliminaries

Let $\tilde{M}$ be an m-dimensional Lorentzian almost paracontact manifold equipped with an almost paracontact structure (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a characteristic vector field, η is a 1-form and g is a Riemannian metric satisfying the following conditions [15]:

$$\phi^2 X_1 = X_1 + \eta(X_1)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.2}$$

$$g(\phi X_1, X_2) = g(X_1, \phi X_2), \quad \eta(X_1) = g(X_1, \xi), \tag{2.3}$$

for all vector fields X_1 and X_2 on $\tilde{M}$. Then the structure (ϕ, ξ, η, g) is said to be Lorentzian para-contact structure.

A Lorentzian paracontact manifold $\tilde{M}$ is called a Lorentzian para-Sasakian(LP-Sasakian) manifold if [15]

$$(\tilde{\nabla}_{X_1}\phi)X_2 = g(X_1, X_2)\xi + \eta(X_2)X_1) + 2\eta(X_1)\eta(X_2)\xi, \tag{2.4}$$

$$\tilde{\nabla}_{X_1}\xi = \phi X_1, \tag{2.5}$$

for all vector fields X_1, X_2 tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection associated with the Riemannian metric g. Furthermore, an almost contact metric manifold $\tilde{M}$ with structure (ϕ, ξ, η, g) is called nearly Lorentzian para-Sasakian if [20]

$$(\tilde{\nabla}_{X_1}\phi)X_2 + (\tilde{\nabla}_{X_2}\phi)X_1 = 2g(X_1, X_2)\xi + +4\eta(X_1)\eta(X_2)\xi + \eta(X_1)X_2 + \eta(X_2)X_1. \tag{2.6}$$

The covariant derivative of the tensor field ϕ is given by

$$(\tilde{\nabla}_{X_1}\phi)X_2 = \tilde{\nabla}_{X_1}\phi X_2 - \phi\tilde{\nabla}_{X_1}X_2. \tag{2.7}$$

Let N be an n-dimensional submanifold immersed in $\tilde{M}$ with induced metric g. The tangent and normal subspaces of N in $\tilde{M}$ are denoted by $\Gamma(TN)$ and $\Gamma(T^{\perp}N)$, respectively.

The Gauss and Weingarten formulas are given by [8]:

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + h(X_1, X_2) \tag{2.8}$$

and

$$\tilde{\nabla}_{X_1}X_5 = -A_{X_5}X_1 + \nabla^{\perp}_{X_1}X_5, \tag{2.9}$$

for all $X_1,\ X_2 \in \Gamma(TN)$ and $X_5 \in \Gamma(T^{\perp}N)$, where ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent bundle TN and $T^{\perp}N$ of N, respectively.

The second fundamental form h and shape operator A are related by the following equation [8]:

$$g(A_{X_5}X_1, X_2) = g(h(X_1, X_2), X_5), \tag{2.10}$$

for any $X_1,\ X_1 \in \Gamma(TN)$ and $X_1 \in \Gamma(T^{\perp}N)$. The mean curvature vector H of N is given by

$$H = \frac{1}{n}tr(\sigma) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.11}$$

where n is the dimension of N and $(e_1, e_2, ..., e_n)$ is an local orthonormal frame of N.

A submanifold N is said to be totally umbilical if

$$h(X_1, X_2) = g(X_1, X_2)H, \tag{2.12}$$

for any $X_1,\ X_2 \in \Gamma(TN)$.

A submanifold N is said to be totally geodesic if $h(X_1, X_2) = 0$ and N is said to be minimal if $H = 0$.

Additionally, we define [18]

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad i, j = 1, 2, .., n, \quad r = n+1, ..., 2m+1,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For a differential function f on an n-dimensional manifold N, the gradient ∇f of f is defined by [8]

$$g(\nabla f, X_1) = X_1 f, \tag{2.13}$$

for any $X_1 \in \Gamma(TN)$. Consequently, for an orthonormal frame $\{e_1, e_2, ..., e_n\}$, we have [8]

$$\|\nabla f\|^2 = \sum_{i=1}^{n}(e_i(f))^2. \tag{2.14}$$

For any $X_1 \in \Gamma(TN)$, we can express ϕX_1 as

$$\phi X_1 = TX_1 + NX_1, \tag{2.15}$$

where TX_1 and NX_1 represent the tangential and normal components of ϕX_1 respectively.

Similarly, for any vector $X_5 \in \Gamma(T^{\perp}N)$, we have

$$\phi X_5 = tX_5 + nX_5,$$

where tX_5 and nX_5 denote the tangential and normal components of ϕX_5, respectively.

Furthermore, using (2.3) and (2.15), we get that

$$g(TX_1, X_2) = g(X_1, TX_2),$$

for any vector $X_1, X_2 \in \Gamma(T^{\perp}N)$. For submanifolds tangent to the structure vector field ξ, there are various classes of submanifolds. We highlight the following [5]:

(1) A submanifold N tangent to ξ is called an invariant submanifold if ϕ preserves every tangent space of N, that is, $\phi(T_pN) \subseteq T_pN$, for every $p \in N$.

(2) A submanifold N tangent to ξ is termed an anti-invariant submanifold if ϕ maps every tangent space of N into the normal space, that is, $\phi(T_pN) \subseteq T_p^{\perp}N$, for every $p \in N$.

(3) A submanifold N of a Lorentzian almost paracontact manifold $\tilde{M}$, tangent to ξ, is referred to as a contact CR-submanifold if there exists a differential distribution D on N such that its orthogonal complementary distribution $D^{\perp}$ is anti-invariant. Specifically, the following conditions must be satisfied:

   (i) $TN = D \oplus D^{\perp} \oplus \langle\xi\rangle$,

   (ii) D is an invariant distribution, i.e., $\phi D \subseteq TN$,

   (iii) $D^{\perp}$ is an anti-invariant distribution, i.e., $\phi D^{\perp} \subseteq T^{\perp}N$.

(4) ([13]) A submanifold N of a Lorentzian almost paracontact manifold $\tilde{M}$ is said to be a slant submanifold if for every point $x \in N$ and any vector $X_1 \in T_xN$ the Wirtinger angle, defined as the angle between ϕX_1 and TX_1, remains constant and is denoted by $\theta \in [0, 2\pi]$. In this case, θ is referred to as the slant angle of N in $\tilde{M}$.

   The invariant and anti-invariant submanifolds are special cases of slant submanifolds, where $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold that is neither invariant nor anti-invariant is referred to as a proper slant submanifold.

The following provides a useful characterization of slant submanifolds in a Lorentzian almost paracontact manifold.

Theorem 2.1. *([13]) Let N be a submanifold of a Lorentzian almost paracontact metric manifold $\tilde{M}$ with $\xi \in \Gamma(TN)$. Then, N is a slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{2.16}$$

Furthermore, if θ is the slant angle of N, then $\lambda = cos^2\theta$.

Thus, the following are the consequences of the above theorem:

$$g(TX_1, TX_2) = cos^2\theta(g(X_1, X_2) + \eta(X_1)\eta(X_2)), \tag{2.17}$$

$$g(NX_1, NX_2) = sin^2\theta(g(X_1, X_2) + \eta(X_1)\eta(X_2)), \tag{2.18}$$

for any $X_1, X_2 \in \Gamma(TN)$.

Definition 2.1. ([11]) A CR-slant warped product $N = (N_T \times N_\perp) \times_f N_\theta$ is defined as $D_1 \oplus D_2$-mixed totally geodesic if $h(D_1, D_2) = 0$, where D_1, D_2 are distributions belonging to $\{D_T, D_\perp, D_\theta\}$.

Definition 2.2. A submanifold N of a Lorentzian almost paracontact metric manifold $\tilde{M}$, tangent to ξ, is called a CR-slant warped product if it can be expressed as a warped product of the form $N = B \times_f N_\theta$, where the fiber N_θ is a proper slant and the base $B = N_T \times N_\perp$ is the CR-product of invariant and anti-invariant submanifolds of $\tilde{M}$.

The tangent bundle of the CR-slant warped product submanifold is decomposed as follows:

$$TN = D_T \oplus D_\perp \oplus D_\theta \oplus \langle\xi\rangle, \tag{2.19}$$

where D_T is an invariant distribution, $D_\perp$ is an anti-invariant distribution and D_θ is a proper slant distribution and $\langle\xi\rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ.

Furthermore, if ν is the ϕ-invariant subspace of the normal bundle $T^\perp N$, then for a CR-slant warped product submanifold, the normal bundle $T^\perp N$ can be decomposed as follows:

$$T^\perp N = \phi D^\perp \oplus FD_\theta \oplus \nu.$$

3. CR-slant warped product submanifolds of a nearly Lorentzian para-Sasakian manifold

In this section, we examine CR-slant warped product submanifold of a nearly Lorentzian para-Sasakian manifold $\tilde{M}$ of the form $N = B \times_f N_\theta$, where $B = N_T \times N_\perp$, is a CR-product of invariant and anti-invairant submanifolds of $\tilde{M}$, N_θ is a slant submanifold and f denotes the warping function. We derive an inequality for this type of submanifold. Throughtout this paper, we denote the corresponding tangent spaces of N_T, $N_\perp$ and N_θ by D_T, $D_\perp$ and D_θ, respectively.

First, we present the following results for future reference.

Lemma 3.1. *Let $N = B \times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Lorentzian para-Sasakian manifold $\tilde{M}$ where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. Then, for any vector fields $X_1, X_2 \in \Gamma(TN_T)$, $X_3, X_4 \in \Gamma(TN_\perp)$ and $X_5 \in \Gamma(TN_\theta)$, we have* *(i)* $\xi(lnf) = 0$,

(ii) $g(h(X_1, X_3), NX_5) = -g(h(X_1, X_5), \phi X_3)$,

(iii) $g(h(X_3, X_4), NX_5) = -3g(h(X_5, X_4), \phi X_3)$,

(iv) $g(h(X_1, X_2), NX_5) = 0$.

Proof. To prove part (i), we consider, $X_5 \in \Gamma(TN_\theta)$ and note that ξ is tangent to B. Then, by equations (2.6) and (2.7), we obtain:

$$\tilde{\nabla}_{X_5}\phi\xi - \phi\tilde{\nabla}_{X_5}\xi + \tilde{\nabla}_\xi\phi X_5 - \phi\tilde{\nabla}_\xi X_5 = -X_5. \tag{3.1}$$

Making use of (2.1), (2.8), and (1.1) in (3.1), we obtain

$$-2\phi h(X_5, \xi) - \xi(lnf)\phi X_5 = -X_5. \tag{3.2}$$

Contracting (3.2) with ϕX_5, we obtain

$$\xi(lnf) = 0.$$

Now, we prove part (ii). From (2.6), we obtain

$$(\tilde{\nabla}_{X_3}\phi)X_5 + (\tilde{\nabla}_{X_5}\phi)X_3 = \eta(X_3)X_5,$$

for $X_3 \in \Gamma(TN_\perp)$, $X_5 \in \Gamma(TN_\theta)$.

By using (2.7), (2.15), (2.8) and (1.1) in preceding equation, we get

$$\begin{aligned} -X_3(lnf)TX_5 + h(X_3, TX_5) + \nabla^\perp_{X_3}NX_5 - A_{NX_5}X_3 - 2X_3(lnf)NX_5 \\ - 2\phi h(X_3, X_5) + \nabla^\perp_{X_5}\phi X_3 - A_{\phi X_3}X_5 = \eta(X_3)X_5. \end{aligned} \tag{3.3}$$

Contracting (3.3) with $X_1 \in \Gamma(TN_T)$, we obtain

$$g(A_{NX_5}X_3, X_1) + g(A_{\phi X_3}X_5, X_1) = 0. \tag{3.4}$$

Substituting (2.10) in (3.4), we obtain part (ii).

Likewise, by contracting (3.3) with $X_4 \in \Gamma(TN_\perp)$ and using (2.10), we derive

$$g(h(X_3, X_4), NX_5) + g(h(X_5, X_4), \phi X_3) = -2g(h(X_3, X_5), \phi X_4). \tag{3.5}$$

In (3.5), we use the polarization identity for $X_3, X_4 \in \Gamma(TN_\perp)$ to find

$$g(h(X_3, X_4), NX_5) + 2g(h(X_5, X_4), \phi X_3) = -g(h(X_3, X_5), \phi X_4). \tag{3.6}$$

Part (iii) is obtained from (3.5) and (3.6).

To demonstrate part (iv), we take $X_1, X_2 \in \Gamma(TN_T)$ and $X_5 \in \Gamma(TN_\theta)$. Utilizing (2.8), (1.1), (2.15), (2.3) and (2.7), we find that

$$\begin{aligned} g(h(X_1, X_2), NX_5) &= g(\tilde{\nabla}_{X_1}X_2, NX_5) - g(\nabla_{X_1}X_2, NX_5) \\ &= g(\tilde{\nabla}_{X_1}X_2, \phi X_5 - TX_5) \\ &= g(\phi\tilde{\nabla}_{X_1}X_2, X_5) - g(\nabla_{X_1}X_2, TX_5) \\ &= g(\tilde{\nabla}_{X_1}\phi X_2, X_5) - g((\tilde{\nabla}_{X_1}\phi)X_2, X_5) \\ &= g(\nabla_{X_1}\phi X_2, X_5) - g((\tilde{\nabla}_{X_1}\phi)X_2, X_5) \end{aligned}$$

$$= -g((\tilde{\nabla}_{X_1}\phi)X_2, X_5). \tag{3.7}$$

When we substitute X_1 for X_2, we obtain

$$g(h(X_1, X_2), NX_5) = -g((\tilde{\nabla}_{X_2}\phi)X_1, X_5). \tag{3.8}$$

From (3.7) and (3.8), we get

$$2g(h(X_1, X_2), NX_5) = -g((\tilde{\nabla}_{X_1}\phi)X_2 + (\tilde{\nabla}_{X_2}\phi)X_1, X_5).$$

By substituting (2.6) into the above equation, we derive part (iv). Thus, the proof is complete. □

Lemma 3.2. *Let $N = B \times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Lorentzian para-Sasakian manifold $\tilde{M}$ where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. Then, we have*

$$\begin{aligned} &g(h(X_5, X_6), \phi X_3) \\ &= -\,(3g(h(X_3, X_5), NX_6) + \eta(X_3)g(X_5, X_6) + X_3(lnf)g(TX_5, X_6)), \end{aligned} \tag{3.9}$$

for any $X_3 \in \Gamma(TN_\perp)$ and $X_5, X_6 \in \Gamma(TN_\theta)$.

Proof. Let $X_3 \in \Gamma(TN_\perp)$ and $X_5, X_6 \in \Gamma(TN_\theta)$. By contracting (3.3) with $X_6 \in \Gamma(TN_\theta)$ and applying (2.10), we obtain

$$\begin{aligned} g(h(X_3, X_6), NX_5) + g(h(X_5, X_6), \phi X_3) + 2g(h(X_3, X_5), FX_6) \\ = -(\eta(X_3)g(X_5, X_6) + X_3(lnf)g(TX_5, X_6)). \end{aligned} \tag{3.10}$$

Using the polarization identity for $X_5, X_6 \in \Gamma(TN_\theta)$ in (3.10), we arrive at

$$\begin{aligned} g(h(X_3, X_5), NX_6) + g(h(X_5, X_6), \phi X_3) + 2g(h(X_3, X_6), NX_5) \\ = -(\eta(X_3)g(X_5, X_6) + X_3(lnf)g(TX_5, X_6)). \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we derive (3.9), which concludes the proof of the Lemma. □

From Lemma (3.2), the following equations can be deduced: by replacing X_6 by TX_6 (and X_5 by TX_5) in (3.9) and utilizing (2.16) and (2.17), we obtain, respectively:

$$\begin{aligned} g(h(X_5, TX_6), \phi X_3) = &-(3g(h(X_3, X_5), NTX_6) + \eta(X_3)g(X_5, TX_6) \\ &+ cos^2\theta X_3(lnf)g(X_5, X_6)). \end{aligned} \tag{3.12}$$

$$\begin{aligned} g(h(TX_5, X_6), \phi X_3) = &-(3g(h(X_3, TX_5), NX_6) + \eta(X_3)g(TX_5, X_6) \\ &+ cos^2\theta X_3(lnf)g(X_5, X_6)). \end{aligned} \tag{3.13}$$

Additionally, by replacing X_6 by TX_6 in (3.13) and applying (2.17), we get

$$\begin{aligned} g(h(TX_5, TX_6), \phi X_3) = &-(3g(h(X_3, TX_5), NTX_6) + cos^2\theta\eta(X_3)g(X_5, X_6) \\ &+ cos^2\theta X_3(lnf)g(X_5, TX_6)). \end{aligned} \tag{3.14}$$

From (3.12) and (3.13), it can be observed that:

$$\begin{aligned} g(h(X_5, TX_6), \phi X_3) + g(h(TX_5, X_6), \phi X_3) = &- (3g(h(X_3, TX_5), NX_6) \\ &+ 3g(h(X_3, X_5), NTX_6) \\ &+ 2\eta(X_3)g(TX_5, X_6) \\ &+ 2cos^2\theta X_3(lnf)g(X_5, X_6)). \quad (3.15) \end{aligned}$$

Lemma 3.3. *Let $N = B \times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Lorentzian para-Sasakian manifold $\tilde{M}$ where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(X_5, X_1), NX_6) = \frac{1}{3}(\phi X_1(lnf) - \eta(X_1))g(X_5, X_6) - \frac{1}{3}X_1(lnf)g(TX_5, X_6), \quad (3.16)$$

for any $X_1 \in \Gamma(TN_T)$ and $X_5, X_6 \in \Gamma(TN_\theta)$.

Proof. Let $X_1 \in \Gamma(TN_T)$ and $X_5, X_6 \in \Gamma(TN_\theta)$. Using (2.6), we have

$$(\tilde{\nabla}_{X_1}\phi)X_5 + (\tilde{\nabla}_{X_5}\phi)X_1 = \eta(X_1)X_5. \quad (3.17)$$

By applying (2.7), (2.15), (2.8) and (1.1), the above equation simplifies to:

$$\begin{aligned} &h(X_1, TX_5) + \nabla^\perp_{X_1} NX_5 - A_{NX_5}X_1 + \phi X_1(lnf)X_5 - X_1(lnf)TX_5 \\ &- 2X_1(lnf)NX_5 - 2\phi h(X_1, X_5) + h(X_5, \phi X_1) = \eta(X_1)X_5. \quad (3.18) \end{aligned}$$

Contracting (3.18) with $X_6 \in \Gamma(TN_\theta)$, we derive

$$\begin{aligned} 2g(h(X_1, X_5), NX_6) + g(h(X_1, X_6), NX_5) =&(\phi X_1(lnf) - \eta(X_1))g(X_5, X_6) \\ &- X_1(lnf)g(TX_5, X_6). \quad (3.19) \end{aligned}$$

By interchanging X_5 with X_6 in (3.19), we get

$$\begin{aligned} 2g(h(X_1, X_6), NX_5) + g(h(X_1, X_5), NX_6) =&(\phi X_1(lnf) - \eta(X_1))g(X_5, X_6) \\ &- X_1(lnf)g(TX_6, X_5). \quad (3.20) \end{aligned}$$

From (3.19) and (3.20), we obtain (3.16), thereby proving the Lemma. □

From Lemma (3.3) and using (2.17), we derive the following equations. In particular, by replacing X_1 with ϕX_1, X_5 with TX_5 and X_6 with TX_6 in (3.16), we get

$$g(h(X_5, \phi X_1), NX_6) = \frac{1}{3}X_1(lnf)g(X_5, X_6) - \frac{1}{3}\phi X_1(lnf)g(TX_5, X_6), \quad (3.21)$$

$$g(h(TX_5, X_1), NX_6) = \frac{1}{3}(\phi X_1(lnf) - \eta(X_1))g(TX_5, X_6) - \frac{1}{3}X_1(lnf)cos^2\theta g(X_5, X_6), \quad (3.22)$$

$$g(h(X_5, X_1), NTX_6) = \frac{1}{3}(\phi X_1(lnf) - \eta(X_1))g(X_5, TX_6) - \frac{1}{3}X_1(lnf)cos^2\theta g(X_5, X_6). \quad (3.23)$$

Similarly, by replacing X_5 with TX_5 (and X_6 with TX_6) in (3.21), we obtain

$$g(h(TX_5, \phi X_1), NX_6) = \frac{1}{3}X_1(lnf)g(TX_5, X_6) - \frac{1}{3}\phi X_1(lnf)cos^2\theta g(X_5, X_6), \quad (3.24)$$

$$g(h(X_5,\phi X_1),NTX_6)=\frac{1}{3}X_1(lnf)g(X_5,TX_6)-\frac{1}{3}\phi X_1(lnf)g(X_5,X_6). \quad (3.25)$$

From (3.24) and by replacing X_6 with TX_6, we have

$$g(h(TX_5,\phi X_1),NTX_6)=\frac{1}{3}X_1(lnf)cos^2\theta g(X_5,X_6)-\frac{1}{3}\phi X_1(lnf)cos^2\theta g(X_5,TX_6). \quad (3.26)$$

From (3.23) and by replacing X_5 with TX_5, we obtain

$$\begin{aligned}g(h(TX_5,X_1),NTX_6)=&\frac{1}{3}(\phi X_1(lnf)-\eta(X_1))cos^2\theta g(X_5,X_6)\\&-\frac{1}{3}X_1(lnf)cos^2\theta g(TX_5,X_6). \quad (3.27)\end{aligned}$$

Clearly, from (3.24) and (3.25), and similarly from (3.22) and (3.23), we derive the following, respectively:

$$g(h(TX_5,\phi X_1),NX_6)-g(h(X_5,\phi X_1),NTX_6)=0, \quad (3.28)$$

$$g(h(TX_5,X_1),NX_6)-g(h(X_5,X_1),NTX_6)=0. \quad (3.29)$$

If ξ is tangent to N_T, it can also be observed from (3.16), (3.27) and (3.23) that we have:

$$g(h(X_5,\xi),NX_6)=\frac{1}{3}g(X_5,X_6), \quad (3.30)$$

$$g(h(TX_5,\xi),NTX_6)=\frac{1}{3}cos^2\theta g(X_5,X_6), \quad (3.31)$$

$$g(h(X_5,\xi),NTX_6)=\frac{1}{3}g(X_5,TX_6). \quad (3.32)$$

Theorem 3.1. *Let $N=B\times_f N_\theta$ be a $D_\perp\oplus D_\theta$-mixed geodesic CR-slant warped product submanifold of a nearly Lorentzian para-Sasakian manifold $\tilde{M}$ where $B=N_T\times N_\perp$ is a CR-product submanifold and N_θ is a slant submanifold of $\tilde{M}$. Then, the second fundamental form h of N satisfies the following inequalities:*

(i) If the structure vector field ξ is tangent to N_T, it follows that

$$\|h\|^2\geq 6lcos^2\theta(\|\nabla_\perp(lnf\|^2)+\frac{8l}{9}\left[\left(csc^2\theta+cot^2\theta\right)(\|\nabla_T(lnf)\|^2)+csc^2\theta\right], \quad (3.33)$$

(ii) If the structure vector field ξ is tangent to $N_\perp$, it folllows that

$$\|h\|^2\geq 6lcos^2\theta(\|\nabla_\perp(lnf)\|^2)+\frac{8l}{9}\left(csc^2\theta+cot^2\theta\right)\|\nabla_T(lnf)\|^2, \quad (3.34)$$

where $\nabla_T(lnf)$ and $\nabla_\perp(lnf)$ represent the gradient of the function lnf in the directions of N_T and $N_\perp$, respectively, and $l=\frac{1}{2}dimN_\theta$.

If the equality holds identically in (3.33) and (3.34), then B is totally geodesic submanifold of $\tilde{M}$ and N_θ is a totally umbilical submanifold of $\tilde{M}$, respectively. Furthermore, N is a $D_T\oplus D_\perp$-mixed totally geodesic submanifold of $\tilde{M}$ but not $D_T\oplus D_\theta$-mixed totally geodesic. Therefore, N is not minimal in $\tilde{M}$.

Proof. Let $\tilde{M}$ be a $(2m+1)$-dimensional nearly Lorentzian para-Sasakian manifold and let $N = B \times_f N_\theta$ be an n-dimensional CR-slant warped product submanifold, where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ.

Let us consider $s = \frac{1}{2}dimN_T$, $dimN_\perp = t$ and $l = \frac{1}{2}dimN_\theta$, so that $n = 2s+t+2l$. We define the following local orthonormal frames: on N_T, $\{e_1, e_2, ..., e_s, e_{s+1} = \phi e_1, ..., e_{2s} = \phi e_s, e_{2s+1} = \xi\}$; on $N_\perp$, $\{e_{2s+2} = \tilde{e}_1, ..., e_{2s+t+1} = \tilde{e}_t\}$; and on N_θ, $\{e_1^* = e_{2s+t+2}, ..., e_l^* = e_{2s+t+l+1}, e_{l+1}^* = sec\theta TE_1^*, ..., e_{2l}^* = sec\theta Te_l^*\}$.

The orthonormal frames in the normal bundle $T^\perp N$ of $\phi D^\perp$ are $\{e_{n+1} = \hat{e_1} = \phi\tilde{e}_1, ..., e_{n+t} = \hat{e_t} = \phi\tilde{e}_t\}$, the orthonormal basis ND_θ is $\{e_{n+t+1} = \hat{e}_{t+1} = csc\theta Ne_1^*, ..., e_{n+t+l} = \hat{e}_{t+l} = csc\theta Ne_l^*, ..., e_{n+t+l+1} = \hat{e}_{t+l+1} = csc\theta sec\theta NTe_1^*, ..., e_{n+t+2l} = \hat{e}_{t+2l} = csc\theta sec\theta NTe_l^*\}$, and the invariant normal subbundle ν is $\{e_{n+t+2l+1}, ..., e_{2m+1}\}$, respectively.

According to the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2.$$

For the assumed frames, the above equation can be expressed as

$$\begin{aligned}\|h\|^2 = &\sum_{r=n+1}^{n+t} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+t+1}^{n+t+2l} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 \\ &+ \sum_{r=n+t+2l+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2. \end{aligned} \tag{3.35}$$

The first term on the right hand side of (3.35) represents the $\phi D^\perp$-component, while the second term corresponds to the FD_θ-component and the third term signifies the ν-component. By excluding the third term and decomposing the first two terms on the right-hand side of (3.35) with respect to the orthonormal frame fields of D_T, $D_\perp$ and D_θ, we obtain

$$\begin{aligned}\|h\|^2 \geq &\sum_{r=1}^{t} \sum_{i,j=1}^{2s+1} g(h(e_i, e_j), \phi\tilde{e}_r)^2 + 2\sum_{r=1}^{t} \sum_{i=1}^{2s+1} \sum_{j=1}^{t} g(h(e_i, \tilde{e}_j), \phi\tilde{e}_r)^2 \\ &+ 2\sum_{r=1}^{t} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(e_i, e_j^*), \phi\tilde{e}_r)^2 + \sum_{r=1}^{t} \sum_{i,j=1}^{t} g(h(\tilde{e}_i, \tilde{e}_j), \phi\tilde{e}_r)^2 \\ &+ 2\sum_{r=1}^{t} \sum_{i=1}^{t} \sum_{j=1}^{2l} g(h(\tilde{e}_i, e_j^*), \phi\tilde{e}_r)^2 + \sum_{r=1}^{t} \sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 \\ &+ \sum_{r=t+1}^{t+2l} \sum_{i,j=1}^{2s+1} g(h(e_i, e_j), \hat{e}_r)^2 + 2\sum_{r=t+1}^{t+2l} \sum_{i=1}^{2s+1} \sum_{j=1}^{t} g(h(e_i, \tilde{e}_j), \hat{e}_r)^2 \\ &+ 2\sum_{r=t+1}^{t+2l} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(e_i, e_j^*), \hat{e}_r)^2 + \sum_{r=t+1}^{t+2l} \sum_{i,j=1}^{t} g(h(\tilde{e}_i, \tilde{e}_j), \hat{e}_r)^2 \\ &+ 2\sum_{r=t+1}^{t+2l} \sum_{i=1}^{t} \sum_{j=1}^{2l} g(h(\tilde{e}_i, e_j^*), \hat{e}_r)^2 + \sum_{r=t+1}^{t+2l} \sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \hat{e}_r)^2. \end{aligned} \tag{3.36}$$

The fifth and eleventh terms on the right-hand side of (3.36) vanish identically, as N is $D_\perp \oplus D_\theta$-mixed geodesic. Similarly, the third, seventh, eighth and tenth terms vanish identically by applying Lemma (3.1). Additionally, we were unable to find the relations for warped products of the first, second, fourth and twelth terms in (3.36). As a result, we keep these positive terms and consider them in the equality case. We now proceed to compute the sixth and ninth terms on the right-hand side of (3.36), that is

$$\|h\|^2 \geq \sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 + 2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i, e_j^*), \hat{e}_r)^2. \tag{3.37}$$

Now, we decompose the first term on the right-hand side of (3.37), as follows:

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 = \sum_{r=1}^{t}\sum_{i,j=1}^{l} \Big[& g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 + g(h(sec\theta Te_i^*, e_j^*), \phi\tilde{e}_r)^2 \\ & + g(h(e_i^*, sec\theta Te_j^*), \phi\tilde{e}_r)^2 \\ & + g(h(sec\theta Te_i^*, sec\theta Te_j^*), \phi\tilde{e}_r)^2\Big].\end{aligned} \tag{3.38}$$

Given that N is a $D_\perp \oplus D_\theta$-mixed geodesic and by (3.15), the second and third terms in (3.38) vanish, and since $\eta(\tilde{e}_r) = 0$ for any $r \in \{1, 2, ..., t\}$, it leads to

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 = \sum_{r=1}^{t}\sum_{i,j=1}^{l} [& g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 + sec^4\theta g(h(Te_i^*, Te_j^*), \phi\tilde{e}_r)^2 \\ & + 4lcos^2\theta(\tilde{e}_r(lnf))^2].\end{aligned} \tag{3.39}$$

By substituting (3.9), (3.14) and (2.14) in (3.39), we derive

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 &= 6lcos^2\theta\sum_{r=1}^{t}(\tilde{e}_r(lnf))^2 \\ &= 6lcos^2\theta\|\nabla_\perp(lnf)\|^2.\end{aligned} \tag{3.40}$$

Again, we decompose second term on the right-hand side of (3.37) and using (3.28) and (3.29), we find that

$$\begin{aligned}2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i, e_j^*), \hat{e}_r)^2 = 2\sum_{i=1}^{s}\sum_{r,j=1}^{l} \Big[& g(h(e_i, e_j^*), csc\theta Ne_r^*)^2 \\ & + g(h(e_i, sec\theta Te_j^*), csc\theta sec\theta NTe_r^*)^2 \\ & + g(h(\phi e_i, e_j^*), csc\theta Ne_r^*)^2 \\ & + g(h(\phi e_i, sec\theta Te_j^*), csc\theta sec\theta NTe_r^*)^2 \\ & + 2g(h(\phi e_i, sec\theta Te_j^*), csc\theta Ne_r^*)^2 \\ & + 2g(h(e_i, e_j^*), csc\theta sec\theta NTe_r^*)^2\Big] \\ & + 2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi, e_j^*), \hat{e}_r)^2.\end{aligned} \tag{3.41}$$

By employing (3.16), (3.27), (3.21), (3.26), (3.24), (3.23) in (3.41) and noting that $\eta(e_i) = 0$ for any $i \in \{1, 2, ..., 2s\}$, we have

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i, e_j^*), \hat{e}_r)^2 = \frac{8l}{9}(csc^2\theta + cot^2\theta)\left[\sum_{i=1}^{2s+1}(e_i(lnf))^2 - (\xi(lnf))^2\right] + 2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi, e_j^*), \hat{e}_r)^2. \tag{3.42}$$

We decompose the last term on the right side of (3.42) and using (3.29), yields

$$2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi, e_j^*), \hat{e}_r)^2 = 2\sum_{r=1}^{l}\sum_{j=1}^{l}\Big[g(h(\xi, e_j^*), csc\theta N e_r^*)^2 + g(h(\xi, sec\theta T e_j^*), csc\theta sec\theta NT e_r^*)^2 + 2g(h(\xi, e_j^*), csc\theta sec\theta NT e_r^*)^2\Big].$$

Substituting (3.30), (3.31) and (3.32) in previous equation, we obtain

$$2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi, e_j^*), \hat{e}_r)^2 = \frac{8l}{9}csc^2\theta. \tag{3.43}$$

By utilizing (2.14), part (i) of Lemma (3.1) and (3.43), we can reformulate (3.42) as

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i, e_j^*), \hat{e}_r)^2 = \frac{8l}{9}\left[\left(csc^2\theta + cot^2\theta\right)\|\nabla_T(lnf)\|^2 + csc^2\theta\right]. \tag{3.44}$$

Thus, by substituting (3.40) and (3.44) into (3.37), we obtain (3.33), thereby proving inequality (i).

To prove inequality (ii), let us consider ξ tagent to $N_\perp$. From (3.42) and by applying (2.14), we get

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s}\sum_{j=1}^{2l} g(h(e_i, e_j^*), \hat{e}_r)^2 = \frac{8l}{9}(csc^2\theta + cot^2\theta)\|\nabla_T(lnf)\|^2. \tag{3.45}$$

Additionally, from (3.40) and by using (2.14) and part (i) of Lemma (3.1) we find that

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 &= 6lcos^2\theta\left[\sum_{r=1}^{t+1}(\tilde{e}_r(lnf))^2 - (\xi(lnf))^2\right]\\ &= 6lcos^2\theta\|\nabla_\perp(lnf)\|^2.\end{aligned} \tag{3.46}$$

Thus, from (3.45) and (3.46), we derive (3.34), which proves inequality (ii).

If equality holds in (3.33) and (3.34), then from the omitting third term on the right-hand side of (3.35), we have

$$h(X_1, X_2) \perp \nu, \tag{3.47}$$

for any $X_1, X_2 \in \Gamma(TN)$. In addition, from the omitting first term and vanishing seventh term in (3.36), we find

$$h(D_T, D_T) \perp \phi D_\perp, \quad h(D_T, D_T) \perp FD_\theta. \tag{3.48}$$

Then, from (3.47) and (3.48), we obtain

$$h(D_T, D_T) = \{0\}. \tag{3.49}$$

Similarly, from the omitting fourth term and vanishing tenth term in (3.36), we observe that

$$h(D_\perp, D_\perp) \perp \phi D_\perp, \quad h(D_\perp, D_\perp) \perp FD_\theta. \tag{3.50}$$

Thus, (3.47) and (3.50) yield:

$$h(D_\perp, D_\perp) = \{0\}. \tag{3.51}$$

Furthermore, from the omitting second term and vanishing eighth term in (3.36), we get

$$h(D_T, D_\perp) \perp \phi D_\perp, \quad h(D_T, D_\perp) \perp FD_\theta. \tag{3.52}$$

Then, from (3.47) and (3.52), we obtain

$$h(D_T, D_\perp) = \{0\}. \tag{3.53}$$

Since N is $D_\perp \oplus D_\theta$-mixed geodesic in $\tilde{M}$, we derive

$$h(D_\perp, D_\theta) = \{0\}. \tag{3.54}$$

On the other hand, from the omitting twelth term in (3.36) along with (3.47), we obtain

$$h(D_\theta, D_\theta) \subset \phi D_\perp. \tag{3.55}$$

Likewise, from the vanishing third term in (3.36) with (3.47), we obtain

$$h(D_T, D_\theta) \subset FD_\theta. \tag{3.56}$$

Since B is totally geodesic in N ([6], [9]), this fact, combined with (3.49) and (3.51), implies that B is totally geodesic in $\tilde{M}$. Also, since N_θ is totally umbilical in N ([6], [9]), this fact, together with (3.55) and (3.56), leads to the conclusion that N_θ is totally umbilical in $\tilde{M}$. Furthermore, from (3.53), we can deduce that N is a $D_T \oplus D_\perp$-mixed geodesic submanifold of $\tilde{M}$. However, N can never be a $D_T \oplus D_\theta$-mixed geodesic. Therefore, the theorem is proved. □

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