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A Predator-Prey Biological Model of Multiple Species with Linear Growth Rates

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Abstract The purpose of this paper is to give sufficient conditions for the existence and uniqueness of positive solutions to an elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are super-sub solutions method, eigenvalues of operators, maximum principles, spectrum estimates, inverse function theory, and general elliptic theory. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of species of animals with predator-prey relation in many biological models.

Keywords Predator-prey system, coexistence state, existence, uniqueness, perturbation

MSC(2010) 35J57, 35J67

1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the survival of two or more species of animals in the same environment. Especially, pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the predator-prey model to better understand the competitive interactions between multiple species. Specifically, we investigate the conditions needed for the coexistence of species when the factors affecting them are fixed or perturbed.

2. Literature review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple predator-prey model, commonly known as the Lotka-Volterra predator-prey model. This system describes the predator-prey interaction of two species residing in the same environment in the following manner:

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Suppose two species of animals, one is prey and the other is predator, are residing in a bounded domain Ω. Let $u(x,t)$ be the density of prey and $v(x,t)$ be density of the predator in the place x of Ω at time t. Then we have the following biological interpretation of terms.

(A) The partial derivatives $u_t(x,t)$ and $v_t(x,t)$ mean the rate of change of densities with respect to time t.
(B) The laplacians $\Delta u(x,t)$ and $\Delta v(x,t)$ stand for the diffusion or migration rates.
(C) The rates of self-reproduction of each species of animals are expressed as multiples of some positive constants α, β and current densities $u(x,t), v(x,t)$, i.e. $\alpha u(x,t)$ and $\beta v(x,t)$ which will increase the rate of change of densities in (A), where $\alpha > 0, \beta > 0$ are called the self-reproduction constants.
(D) The rates of self-limitation of each species of animals are multiples of some positive constants a, d and the frequency of encounters among themselves $u^2(x,t)$, $v^2(x,t)$, i.e. $bu^2(x,t)$ and $fv^2(x,t)$ which will decrease the rate of change of densities in (A), where $a > 0, d > 0$ are called the self-limitation constants.
(E) The rates of competition of each species of animals are multiples of some positive constants b, c and the frequency of encounters of each species with the other $u(x,t)v(x,t)$, i.e. $bu(x,t)v(x,t)$ and $cu(x,t)v(x,t)$ which will decrease the rate of change of densities of prey and increase the rate of change of densities of predator in (A), where $b > 0, c > 0$ are called the competition constants.
(F) We assume that none of the species of animals is staying on the boundary of Ω.

Combining all those together, we have the dynamic predator-prey model

$$\begin{cases} \left.\begin{aligned} u_t(x,t) &= \Delta u(x,t) + \alpha u(x,t) - au^2(x,t) - bu(x,t)v(x,t) \\ v_t(x,t) &= \Delta v(x,t) + \beta v(x,t) - dv^2(x,t) + cu(x,t)v(x,t) \end{aligned}\right\} \text{ in } \Omega \times [0,\infty), \\ u(x,t) = v(x,t) = 0 \text{ for } x \in \partial\Omega, \end{cases}$$

or equivalently,

$$\begin{cases} \left.\begin{aligned} u_t(x,t) &= \Delta u(x,t) + u(x,t)(\alpha - au(x,t) - bv(x,t)) \\ v_t(x,t) &= \Delta v(x,t) + v(x,t)(\beta - dv(x,t) + cu(x,t)) \end{aligned}\right\} \text{ in } \Omega \times [0,\infty), \\ u(x,t) = v(x,t) = 0 \text{ for } x \in \partial\Omega. \end{cases}$$

Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x), v(x)$ of

$$\begin{cases} \left.\begin{aligned} \Delta u(x) + u(x)(\alpha - au(x) - bv(x)) = 0 \\ \Delta v(x) + v(x)(\beta - dv(x) + cu(x)) = 0 \end{aligned}\right\} \text{ in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means the two species of animals can live peacefully and forever.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (2.1)([12], [13], [26]) or more generalized population model([8], [9], [10], [11], [14], [15]).

One of the initial important results for the time-independent Lotka-Volterra model was obtained by Korman and Leung. In 1986, they published the following sufficient conditions for the existence of a positive steady state solution to (2.1):

Theorem 2.1 ([12]). *If $ad > bc, \alpha > \frac{ad(\lambda_1 + \frac{b}{d}\beta)}{ad - bc}$ and $\beta > \lambda_1$, then (2.1) has a positive solution.*

Biologically, the conditions in Theorem 2.1 implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, in other words, if members of each species interact strongly among themselves and weakly with members of the other species, then there is a positive steady state solution to (2.1), that is, the two species within the same domain will coexist indefinitely at population densities.

Another important result was obtained by Zhengyuan and Mottoni. In 1992, they published the following characterization of non-negative solutions to (2.1) in terms of growth rates (α, β):

Theorem 2.2 ([26]). *There exist two functions $\gamma_0(\alpha), \mu_0(\beta)$ such that the set S of non-negative solutions to (2.1) is characterized as follows:*

(A) If $\alpha \le \lambda_1, \beta \le \lambda_1$, then $S = \{(0,0)\}$.
(B) If $\alpha \le \lambda_1, \beta > \lambda_1$, then $S = \{(0,0), (0, \theta_{\frac{\beta}{d}})\}$.
(C) If $\alpha > \lambda_1, \beta < \gamma_0(\alpha)$, then $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0)\}$.
(D) If $\lambda_1 < \alpha < \mu_0(\beta), \beta > \lambda_1$, then $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (0, \theta_{\frac{\beta}{d}})\}$.
(E) If $\alpha > \lambda_1, \gamma_0(\alpha) < \beta \le \lambda_1$, then $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (u^+, v^+)\}$, where (u^+, v^+) is a positive solution to (2.1).
(F) If $\beta > \lambda_1, \alpha > \mu_0(\beta)$, then $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (0, \theta_{\frac{\beta}{d}}), (u^+, v^+)\}$.

These results provide insight into the predator-prey interactions of two species operating under the conditions described in the Lotka-Volterra model. In this paper, our research has been focused on the existence and uniqueness of the positive steady state solution of the predator-prey model for arbitrary N species,

$$\begin{cases} (u_i)_t(x,t) = \Delta u_i(x,t) + u_i(x,t)[a_i + \sum_{j=1}^N b_{ij} u_j(x,t)] \\ \text{in } \ \Omega \times R^+, \\ u_i(x,t)|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases}$$

or, equivalently, the positive solution to

$$\begin{cases} \Delta u_i(x) + u_i(x)[a_i + \sum_{j=1}^N b_{ij} u_j(x)] = 0 \quad \text{in } \ \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{2.2}$$

where $a_i, b_{ij}, i, j = 1, ..., N$ designate reproduction, self-limitation and competition rates such that $a_i > 0, i = 1, ..., N$ and

$$b_{ij} \begin{cases} < 0, i = 1, j = 1, ..., N, \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N. \end{cases}$$

The followings are questions raised in the general model with nonlinear growth rates.
Problem 1 : What are the sufficient conditions for existence of positive solutions?
Problem 2 : What are the sufficient conditions for uniqueness of positive solutions?
Problem 3 : What is the effect of perturbation for existence and uniqueness? In our analysis, we focus on the conditions required for the maintenance of the coexistence state of the model when reproduction rates $(a_1, ..., a_N)$ are slightly perturbed. Biologically, our conclusion implies that the species may slightly relax ecologically and yet continue to coexist at unique densities.

In Section 4, we establish sufficient conditions for the existence and non-existence of positive solution of the system that generalizes the Theorems 2.1 and 2.2. We also achieve solution estimates in Section 5 to prove the uniqueness and the invertibility of linearization in Sections 6, 7 and 8, where we investigate the effect of perturbation for existence and uniqueness.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the model with $N = 2$.(See [2], [3].)

3. Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition 3.1. The vector functions $(\bar{u}^1, ..., \bar{u}^N), (\underline{u}^1, ..., \underline{u}^N)$ form a super/sub solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1, ..., u^N) = 0 \text{ in } \Omega, \\ u^i = 0 \text{ on } \partial\Omega, \end{cases}$$

if for $i = 1, ..., N$,

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1, ..., u^{i-1}, \bar{u}^i, u^{i+1}, ..., u^N) \leq 0, \\ \Delta \underline{u}^i + g^i(u^1, ..., u^{i-1}, \underline{u}^i, u^{i+1}, ..., u^N) \geq 0, \end{cases} \text{in } \Omega \text{ for } \underline{u}^j \leq u^j \leq \bar{u}^j, j \neq i,$$

and

$$\begin{cases} \underline{u}^i \leq \bar{u}^i \text{ on } \Omega, \\ \underline{u}^i \leq 0 \leq \bar{u}^i \text{ on } \partial\Omega. \end{cases}$$

Lemma 3.1. *If g^i in the Definition 3.1 are in C^1 and the system admits a super/sub solution pair $(\underline{u}^1, ..., \underline{u}^N), (\bar{u}^1, ..., \bar{u}^N)$, respectively, then there is a solution $(u_1, ..., u_N)$ to the system in Definition 3.1 with $\underline{u}^i \leq u^i \leq \bar{u}^i$ in $\bar{\Omega}$. If*

$$\Delta\bar{u}^i + g^i(\bar{u}^1, ..., \bar{u}^N) \neq 0,$$
$$\Delta\underline{u}^i + g^i(\underline{u}^1, ..., \underline{u}^N) \neq 0$$

in Ω for $i = 1, ..., N$, then $\underline{u}^i < u^i < \bar{u}^i$ in Ω.

Lemma 3.2.

$$\begin{cases} -\Delta u + q(x)u = \lambda u \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain.
(A) The first eigenvalue $\lambda_1(q)$ of (3.1), denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction ϕ_q.
(B) If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.
(C)(Variational Characterization of the first eigenvalue)

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (|\nabla\phi|^2 + q\phi^2)dx}{\int_\Omega \phi^2 dx}.$$

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

Lemma 3.3. *Consider*

$$\begin{cases} \Delta u + uf(u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain.
(A) If $f(0) > \lambda_1$, then the above equation has a unique positive solution. We denote this unique positive solution as θ_f.
(B) If $f(0) \leq \lambda_1$, then $u \equiv 0$ is the only nonnegative solution to the above equation.

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $\alpha > \lambda_1$, we denote θ_α as the unique positive solution of

$$\begin{cases} \Delta u + u(\alpha - u) = 0 \ \ \text{in} \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0. \end{cases}$$

Hence, θ_α is increasing as $\alpha > 0$ is increasing.

Having established these preliminaries, we now commence our investigation of the general predator-prey model.

4. Existence, nonexistence

An elliptic interacting system of N functions with homogeneous boundary condition is

$$\begin{cases} \Delta u_i + u_i(a_i + \sum_{j=1}^{N} b_{ij}u_j) = 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{4.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $a_i > 0, i = 1, ..., N$ are reproduction rates and b_{ij} are self-limitation and competition rates such that

$$b_{ij} \begin{cases} < 0, i = 1, j = 1, ..., N, \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N. \end{cases}$$

We establish the following existence result:

Theorem 4.1. *If for each $i = 2, ..., N$,*
(A) $b_{11}b_{ii} + (N-1)b_{1i}b_{i1} > 0, i = 2, ..., N$,
(B) $a_1 > \frac{b_{11}b_{ii}(\lambda_1 + \frac{(N-1)b_{1i}a_i}{b_{ii}})}{b_{11}b_{ii}+(N-1)b_{1i}b_{i1}}$ and
(C) $a_i > \lambda_1 - (N-2)\inf_{j=2,...,N, j\neq i}\{\bar{u_j}b_{ij}\}, i = 2, ..., N$, where $\bar{u_j}$ is defined below, then (4.1) has a positive solution.

Proof. Let $\underline{u_i} = \gamma_i\omega, i = 1, ..., N, \bar{u_1} = -\frac{a_1}{b_{11}}, \bar{u_i} = -\frac{1}{b_{ii}}(a_i - \frac{a_1 b_{i1}}{b_{11}}), i = 2, ..., N$, where $\gamma_i > 0$ are constants and ω is the eigenfunction of (3.1) with $q(x) \equiv 0$ corresponding to the first eigenvalue λ_1.

Then for all u_i such that $\underline{u_i} \le u_i \le \bar{u_i}, i = 2, ..., N$,

$$\begin{aligned} &\Delta\bar{u_1} + \bar{u_1}(a_1 + b_{11}\bar{u_1} + \textstyle\sum_{j=2}^{N} b_{1j}u_j) \\ &= \bar{u_1}\textstyle\sum_{j=2}^{N} b_{1j}u_j \\ &< 0, \end{aligned}$$

and for all u_1 such that $\underline{u_1} \le u_1 \le \bar{u_1}$ and for all $i = 2, ..., N$,

$$\begin{aligned} &\Delta\bar{u_i} + \bar{u_i}[a_i + b_{ii}\bar{u_i} + \textstyle\sum_{j=1, j\neq i}^{N} b_{ij}u_j] \\ &\le \bar{u_i}[a_i + b_{i1}\bar{u_1} + b_{ii}\bar{u_i}] \\ &= 0. \end{aligned}$$

By the condition again, for all u_i such that $\underline{u_i} \le u_i \le \bar{u}_i, i = 2, ..., N$,

$$\begin{aligned}
&\Delta \underline{u_1} + \underline{u_1}(a_1 + b_{11}\underline{u_1} + \textstyle\sum_{i=2}^N b_{1i}u_i) \\
&= \Delta(\gamma_1\omega) + \gamma_1\omega(a_1 + b_{11}\gamma_1\omega + \textstyle\sum_{i=2}^N b_{1i}u_i) \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega(a_1 + b_{11}\gamma_1\omega + \textstyle\sum_{i=2}^N b_{1i}u_i) \\
&\ge -\gamma_1\lambda_1\omega + \gamma_1\omega[a_1 + (N-1)\inf_{i=2,...,N}\{b_{1i}\bar{u}_i\} + b_{11}\gamma_1\omega] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega[a_1 + (N-1)\inf_{i=2,...,N}\{-\tfrac{1}{b_{ii}}(a_i - \tfrac{a_1b_{i1}}{b_{11}})b_{1i}\} + b_{11}\gamma_1\omega] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega[a_1 + (N-1)\inf_{i=2,...,N}\{b_{1i}(-\tfrac{a_i}{b_{ii}} + \tfrac{a_1b_{i1}}{b_{11}b_{ii}})\} + b_{11}\gamma_1\omega] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega[\inf_{i=2,...,N}\{a_1 - \tfrac{(N-1)b_{1i}a_i}{b_{ii}} + \tfrac{(N-1)a_1b_{1i}b_{i1}}{b_{11}b_{ii}}\} + b_{11}\gamma_1\omega] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega[\inf_{i=2,...,N}\{a_1(1 + \tfrac{(N-1)b_{1i}b_{i1}}{b_{1i}b_{ii}}) - \tfrac{(N-1)b_{1i}a_i}{b_{ii}}\} + b_{11}\gamma_1\omega] \\
&= \gamma_1\omega[-\lambda_1 + \inf_{i=2,...,N}\{a_1(1 + \tfrac{(N-1)b_{1i}b_{i1}}{b_{11}b_{ii}}) - \tfrac{(N-1)b_{1i}a_i}{b_{ii}}\} + b_{11}\gamma_1\omega] \\
&> 0
\end{aligned}$$

with small enough $\gamma_1 > 0$, and for all $i = 2, ..., N$, $j = 1, ..., N, j \ne i$ and u_j such that $\underline{u_j} \le u_j \le \bar{u}_j$,

$$\begin{aligned}
&\Delta \underline{u_i} + \underline{u_i}(a_i + b_{ii}\underline{u_i} + \textstyle\sum_{j=1,j\ne i}^N b_{ij}u_j) \\
= \;& -\gamma_i\lambda_1\omega + \gamma_i\omega(a_i + b_{ii}\gamma_i\omega + \textstyle\sum_{j=1,j\ne i}^N b_{ij}u_j) \\
\ge \;& \gamma_i\omega[a_i - \lambda_1 + b_{ii}\gamma_i\omega + \textstyle\sum_{j=2,j\ne i}^N b_{ij}\bar{u}_j] \\
> \;& 0
\end{aligned}$$

with small enough $\gamma_i > 0$. Furthermore,

$$\underline{u_i} = \bar{u}_i = 0 \text{ on } \partial\Omega$$

and

$$\underline{u_i} \le \bar{u}_i$$

with small enough $\gamma_i > 0$. Hence, by the Lemma 3.1, there is a solution $(u_1, ..., u_N)$ to (4.1) with

$$\underline{u_i} \le u_i \le \bar{u}_i.$$

□

We also establish the following nonexistence results.

Theorem 4.2. *Suppose $a_i \le \lambda_1, i = 1, ..., N$.*
Then $u_i \equiv 0, i = 1, ..., N$ is the only nonnegative solution to (4.1).

Proof. Let $(u_1, ..., u_N)$ be a nonnegative solution to (4.1). Then, for $i = 2, ..., N$,

$$\begin{aligned}
&\Delta u_1 + u_1(a_1 + b_{11}u_1 + b_{1i}u_i) \ge 0 \text{ in } \Omega, \\
&\Delta u_i + u_i(a_i + b_{ii}u_i + b_{i1}u_1) \ge 0 \text{ in } \Omega.
\end{aligned}$$

Therefore, for $i = 2, ..., N$,

$$\begin{aligned}
&b_{i1}\phi_1\Delta u_1 + b_{i1}\phi_1 u_1(a_1 + b_{11}u_1 + b_{1i}u_i) \ge 0 \text{ in } \Omega, \\
&-b_{1i}\phi_1\Delta u_i - b_{1i}\phi_1 u_i(a_i + b_{ii}u_i + b_{i1}u_1) \ge 0 \text{ in } \Omega.
\end{aligned}$$

So, for $i = 2, ..., N$,

$$\int_\Omega -b_{i1}\phi_1 \Delta u_1 dx \le \int_\Omega [(a_1 + b_{11}u_1)b_{i1}u_1 + b_{1i}b_{i1}u_1u_i]\phi_1 dx,$$
$$\int_\Omega b_{1i}\phi_1 \Delta u_i dx \le \int_\Omega [(-a_i - b_{ii}u_i)b_{1i}u_i - b_{1i}b_{i1}u_1u_i]\phi_1 dx.$$

Hence, by the Green's Identity, we have

$$\int_\Omega b_{i1}\lambda_1\phi_1 u_1 dx \le \int_\Omega [(a_1 + b_{11}u_1)b_{i1}u_1 + b_{1i}b_{i1}u_1u_i]\phi_1 dx,$$
$$\int_\Omega -b_{1i}\lambda_1\phi_1 u_i dx \le \int_\Omega [(-a_i - b_{ii}u_i)b_{1i}u_i - b_{1i}b_{i1}u_1u_i]\phi_1 dx.$$

Therefore,

$$\int_\Omega b_{i1}(\lambda_1 - a_1 - b_{11}u_1)u_1\phi_1 - b_{1i}(\lambda_1 - a_i - b_{ii}u_i)u_i\phi_1 dx \le 0.$$

Since the left hand side is nonnegative from

$$a_1 + b_{11}u_1 \le a_1 \le \lambda_1,$$
$$a_i + b_{ii}u_i \le a_i \le \lambda_1,$$

we conclude that $u_i \equiv 0, i = 1, ..., N$. □

Theorem 4.3. *Let $u_i \ge 0, i = 1, ..., N$ be a solution to (4.1). If $a_1 \le \lambda_1$, then $u_1 \equiv 0$.*

Proof. Proceeding as in the proof of Theorem 4.2, we obtain

$$0 \le \int_\Omega (\lambda_1 - a_1 - b_{11}u_1)u_1\phi_1 dx \le \int_\Omega b_{1i}u_1u_i\phi_1 dx \le 0,$$

and so, $u_1 \equiv 0$. □

5. Solution estimate

In order to prove further results, we will need the following solution estimate.

Lemma 5.1. *Let $(u_1, ..., u_N), u_i \ge 0, i = 1, ..., N$ be a solution of the problem*

$$\begin{cases} -\Delta u_i = tu_i(a_i + \sum_{j=1}^N b_{ij}u_j) \ \ in \ \ \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{5.1}$$

where $t \in [0, 1]$. Then

(A)

$$u_1 \le M_1, u_i \le M_i,$$

where $M_1 = -\frac{a_1}{b_{11}}, M_i = \frac{b_{i1}M_1 + a_i}{-b_{ii}}, i = 2, ..., N$.

(B) For $t = 1$, if $u_i > 0$, $b_{11}b_{ii} + b_{1i}b_{i1} > 0$, $a_1 + \sum_{j=2,j\ne i}^N b_{1j}M_j + b_{1i}[\frac{a_1b_{i1} - a_ib_{11}}{b_{11}b_{ii}}] > \lambda_1$, and $a_i + \sum_{j=2,j\ne i}^N b_{ij}M_j > \lambda_1, i = 2, ..., N$, then

$$-\frac{1}{b_{11}}\theta_{a_1 + \sum_{j=2,j\ne i}^N b_{1j}M_j + b_{1i}\frac{a_1b_{i1} - a_ib_{11}}{b_{11}b_{ii}}} \le u_1 \le -\frac{1}{b_{11}}\theta_{a_1},$$
$$-\frac{1}{b_{ii}}\theta_{a_i + \sum_{j=2,j\ne i}^N b_{ij}M_j} \le u_i \le -\frac{1}{b_{ii}}\theta_{a_i - \frac{a_1b_{i1}}{b_{11}}}, i = 2, ..., N.$$

Proof. (A) Since $(u_1, ..., u_N)$ is a solution to (5.1),

$$\Delta u_1 + u_1(a_1 + b_{11}u_1) \geq -u_1 \sum_{i=2}^{N} b_{1i}u_i \geq 0.$$

Hence, by the Maximum Principles,

$$a_1 + b_{11}u_1 \geq 0,$$

equivalently,

$$u_1 \leq -\frac{a_1}{b_{11}}.$$

Since $(u_1, ..., u_N)$ is a solution to (5.1), by the above estimation, for $i = 2, ..., N$,

$$\begin{aligned}
&\Delta u_i + u_i(a_i + b_{ii}u_i + b_{i1}(-\tfrac{a_1}{b_{11}})) \\
\geq\ &\Delta u_i + u_i(a_i + b_{ii}u_i + b_{i1}u_1) \\
\geq\ &\Delta u_i + u_i(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) \\
=\ &0.
\end{aligned}$$

Hence, by the Maximum Principles again,

$$a_i + b_{ii}u_i + b_{i1}(-\frac{a_1}{b_{11}}) \geq 0,$$

equivalently,

$$u_i \leq \frac{b_{i1}(-\frac{a_1}{b_{11}}) + a_i}{-b_{ii}}, i = 2, ..., N.$$

(B)

$$\Delta u_1 + u_1(a_1 + b_{11}u_1) = u_1(a_1 + b_{11}u_1) - u_1(a_1 + \sum_{j=1}^{N} b_{1j}u_j) \geq 0,$$

and so, u_1 is a subsolution to

$$\begin{aligned}
&\Delta Z + Z(a_1 + b_{11}Z) = 0 \text{ in } \Omega, \\
&Z|_{\partial\Omega} = 0.
\end{aligned}$$

But, since any sufficiently large positive constant is a supersolution to

$$\begin{aligned}
&\Delta Z + Z(a_1 + b_{11}Z) = 0 \text{ in } \Omega, \\
&Z|_{\partial\Omega} = 0,
\end{aligned}$$

by the Lemmas 3.1 and 3.3, we conclude that

$$u_1 \leq -\tfrac{1}{b_{11}}\theta_{a_1}. \tag{5.2}$$

Since

$$\begin{aligned}
&\Delta u_1 + u_1[a_1 + b_{11}u_1 + \textstyle\sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] \\
&= u_1[-a_1 - \textstyle\sum_{j=1}^N b_{1j}u_j + a_1 + b_{11}u_1 + \sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] \\
&\leq u_1[-a_1 - b_{11}u_1 - b_{1i}u_i - \textstyle\sum_{j=2,j\neq i}^N b_{1j}u_j + a_1 + b_{11}u_1 + \sum_{j=2,j\neq i}^N b_{1j}M_j \\
&\quad + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] \\
&\leq u_1[-b_{1i}u_i + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] \\
&= -u_1 b_{1i}[u_i - \frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}}] \\
&\leq 0, i = 2, ..., N,
\end{aligned}$$

by (A), u_1 is a supersolution to

$$\begin{aligned}
&\Delta Z + Z[a_1 + b_{11}Z + \textstyle\sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] = 0 \text{ in } \Omega, \\
&Z|_{\partial\Omega} = 0, i = 2, ..., N.
\end{aligned}$$

But, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&\Delta \epsilon\phi_1 + \epsilon\phi_1[a_1 + b_{11}\epsilon\phi_1 + \textstyle\sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] \\
&= \epsilon\phi_1[-\lambda_1 + a_1 + b_{11}\epsilon\phi_1 + \textstyle\sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] \\
&> 0, i = 2, ..., N,
\end{aligned}$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\begin{aligned}
&\Delta Z + Z[a_1 + b_{11}Z + \textstyle\sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})] = 0 \text{ in } \Omega, \\
&Z|_{\partial\Omega} = 0, i = 2, ..., N.
\end{aligned}$$

Therefore, by the Lemmas 3.1 and 3.3, we have

$$-\frac{1}{b_{11}}\theta_{a_1 + \sum_{j=2,j\neq i}^N b_{1j}M_j + b_{1i}(\frac{b_{i1}[-\frac{a_1}{b_{11}}]+a_i}{-b_{ii}})} \leq u_1, i = 2, ..., N. \tag{5.3}$$

By (A),

$$\begin{aligned}
&\Delta u_i + u_i(a_i + b_{ii}u_i + \textstyle\sum_{j=2,j\neq i}^N b_{ij}M_j) \\
&= u_i(a_i + b_{ii}u_i + \textstyle\sum_{j=2,j\neq i}^N b_{ij}M_j - a_i - \sum_{j=1}^N b_{ij}u_j) \\
&< 0, i = 2, ..., N,
\end{aligned}$$

and so, u_i is a supersolution to

$$\begin{aligned}
&\Delta Z + Z(a_i + b_{ii}Z + \textstyle\sum_{j=2,j\neq i}^N b_{ij}M_j) = 0 \text{ in } \Omega, \\
&Z|_{\partial\Omega} = 0, i = 2, ..., N.
\end{aligned}$$

But, by the condition, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&\Delta \epsilon\phi_1 + \epsilon\phi_1(a_i + b_{ii}\epsilon\phi_1 + \textstyle\sum_{j=2,j\neq i}^N b_{ij}M_j) \\
&= \epsilon\phi_1(-\lambda_1 + a_i + b_{ii}\epsilon\phi_1 + \textstyle\sum_{j=2,j\neq i}^N b_{ij}M_j) \\
&> 0, i = 2, ..., N,
\end{aligned}$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\Delta Z + Z(a_i + b_{ii}Z + \sum_{j=2,j\neq i}^{N} b_{ij}M_j) = 0 \text{ in } \Omega,$$
$$Z|_{\partial\Omega} = 0, i = 2, ..., N.$$

Hence, by the Lemmas 3.1 and 3.3, we have

$$-\frac{1}{b_{ii}}\theta_{a_i+\sum_{j=2,j\neq i}^{N} b_{ij}M_j} \leq u_i, i = 2, ..., N. \tag{5.4}$$

Since $(u_1, ..., u_N)$ is a solution to (5.1), by (A),

$$\begin{aligned}
&\Delta u_i + u_i(a_i + b_{ii}u_i - \tfrac{b_{i1}a_1}{b_{11}}) \\
&= u_i(-a_i - \textstyle\sum_{j=1}^{N} b_{ij}u_j + a_i + b_{ii}u_i - \tfrac{b_{i1}a_1}{b_{11}}) \\
&= u_i(-b_{i1}u_1 - \textstyle\sum_{j=2,j\neq i}^{N} b_{ij}u_j - \tfrac{b_{i1}a_1}{b_{11}}) \\
&\geq u_i(-b_{i1}u_1 - \tfrac{b_{i1}a_1}{b_{11}}) \\
&= b_{i1}u_i(-u_1 - \tfrac{a_1}{b_{11}}) \\
&\geq 0, i = 2, ..., N.
\end{aligned}$$

Therefore, u_i is a subsolution to

$$\Delta Z + Z(a_i + b_{ii}Z - \tfrac{b_{i1}a_1}{b_{11}}) = 0 \text{ in } \Omega,$$
$$Z|_{\partial\Omega} = 0, i = 2, ..., N.$$

But, since any sufficiently large constant is a supersolution to

$$\Delta Z + Z(a_i + b_{ii}Z - \tfrac{b_{i1}a_1}{b_{11}}) = 0 \text{ in } \Omega,$$
$$Z|_{\partial\Omega} = 0, i = 2, ..., N,$$

by the Lemmas 3.1 and 3.3, we have

$$u_i \leq -\frac{1}{b_{ii}}\theta_{a_i - \frac{b_{i1}a_1}{b_{11}}}, i = 2, ..., N. \tag{5.5}$$

By (5.2), (5.3), (5.4) and (5.5), we establish the desired inequalities. □

6. Uniqueness

In this section, we prove the uniqueness of positive solution to (4.1).

We have the following uniqueness result.

Theorem 6.1. *In addition to the theorem 4.1, if*
(A) $a_1 + \sum_{j=2,j\neq i}^{N} b_{1j}M_j + b_{1i}[\frac{a_1b_{i1}-a_ib_{11}}{b_{11}b_{ii}}] > \lambda_1,\ a_i + \sum_{j=2,j\neq i}^{N} b_{ij}M_j > \lambda_1, i = 2, ..., N$*, and*
(B) $2b_{11} + \sum_{i=2}^{N} b_{i1} \sup \frac{-\frac{1}{b_{ii}}\theta_{a_i - \frac{a_1b_{i1}}{b_{11}}}}{-\frac{1}{b_{11}}\theta_{a_1+\sum_{j=2,j\neq i}^{N} b_{1j}M_j + b_{1i}\frac{a_1b_{i1}-a_ib_{11}}{b_{11}b_{ii}}}} < \sum_{j=2}^{N}(b_{1j}$

$+ b_{j1} \inf \frac{-\frac{1}{b_{jj}}\theta_{a_j + \sum_{k=2,k\neq j}^{N} b_{jk}M_k}}{-\frac{1}{b_{11}}\theta_{a_1}})$, *and*

$2b_{ii} + b_{i1} < \sum_{j=2,j\neq i}^{N}(b_{ij} + b_{ji} \sup \frac{-\frac{1}{b_{jj}}\theta_{a_j - \frac{a_1 b_{j1}}{b_{11}}}}{-\frac{1}{b_{ii}}\theta_{a_i + \sum_{k=2,k\neq i}^{N} b_{ik}M_k}})$ *for* $i = 2, ..., N$, *then (4.1) has a unique posotive solution.*

The conditions imply that species 1 interacts strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1.

Proof. The existence was already proved in the last section. We prove the uniqueness.

Let $(u_1, ..., u_N), (v_1, ..., v_N)$ be positive solutions to (4.1), and let $p_i = u_i - v_i, i = 1, ..., N$. We want to show that $p_i \equiv 0, i = 1, ..., N$.

Since $(u_1, ..., u_N), (v_1, ..., v_N)$ are solutions to (4.1),

$$\begin{aligned} &\Delta p_i + p_i(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) \\ &= \Delta u_i - \Delta v_i + (u_i - v_i)(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) \\ &= -\Delta v_i - v_i(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) \\ &= -v_i(\textstyle\sum_{j=1}^{N} b_{ij}u_j - \sum_{j=1}^{N} b_{ij}v_j), i = 1, ..., N. \end{aligned}$$

So,

$$\Delta p_i + p_i(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) - v_i(\sum_{j=1}^{N} b_{ij}v_j - \sum_{j=1}^{N} b_{ij}u_j) = 0, i = 1, ..., N.$$

So, for $i = 1, ..., N$,

$$-p_i\Delta p_i - (p_i)^2(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) + v_ip_i(\sum_{j=1}^{N} b_{ij}v_j - \sum_{j=1}^{N} b_{ij}u_j) = 0.$$

Since

$$\Delta u_i + u_i(a_i + \textstyle\sum_{j=1}^{N} b_{ij}u_j) = 0, i = 1, ..., N,$$

by the Lemma 3.2, we have

$$\textstyle\int_\Omega -p_i\Delta p_i - (a_i + \sum_{j=1}^{N} b_{ij}u_j)(p_i)^2 dx \geq 0, i = 1, ..., N,$$

and so,

$$\int_\Omega v_ip_i(\sum_{j=1}^{N} b_{ij}v_j - \sum_{j=1}^{N} b_{ij}u_j)dx \leq 0, i = 1, ..., N,$$

and so,

$$\int_\Omega \sum_{i=1}^{N} v_ip_i \sum_{j=1}^{N} b_{ij}(-p_j)dx \leq 0.$$

Hence,

$$\textstyle\int_\Omega \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij}(-v_ip_ip_j)dx \leq 0.$$

So,

$$\textstyle\int_\Omega \sum_{i=1}^{N}\{b_{ii}[-v_i(p_i)^2] + \sum_{j=1,j\neq i}^{N} b_{ij}(-v_ip_ip_j)\}dx \leq 0.$$

So,

$$\int_\Omega b_{11}[-v_1(p_1)^2] + \sum_{j=2}^N b_{1j}(-v_1p_1p_j) + \sum_{i=2}^N \{b_{ii}[-v_i(p_i)^2] + \sum_{j=1,j\neq i}^N b_{ij}(-v_ip_ip_j)\}dx \leq 0,$$

so,

$$\int_\Omega b_{11}[-v_1(p_1)^2] + \sum_{j=2}^N b_{1j}(-v_1p_1p_j) + \sum_{i=2}^N \{b_{ii}[-v_i(p_i)^2] + \sum_{j=2,j\neq i}^N b_{ij}(-v_ip_ip_j) + b_{i1}(-v_ip_ip_1)\}dx \leq 0.$$

But, since $b_{i1}(-v_i) < 0$ and $p_ip_1 \leq \frac{(p_i)^2}{2} + \frac{(p_1)^2}{2}$ for $i = 2, ..., N$,

$$\int_\Omega b_{11}[-v_1(p_1)^2] + \sum_{j=2}^N b_{1j}(-v_1p_1p_j) + \sum_{i=2}^N \{b_{ii}[-v_i(p_i)^2] + \sum_{j=2,j\neq i}^N b_{ij}(-v_ip_ip_j) + b_{i1}(-v_i)(\frac{(p_i)^2}{2} + \frac{(p_1)^2}{2})\}dx \leq 0.$$

Hence,

$$\int_\Omega [b_{11}(-v_1) + \sum_{i=2}^N \frac{b_{i1}(-v_i)}{2}](p_1)^2 + \sum_{j=2}^N b_{1j}(-v_1p_1p_j) + \sum_{i=2}^N \{[b_{ii}(-v_i) + \frac{b_{i1}(-v_i)}{2}][(p_i)^2] + \sum_{j=2,j\neq i}^N b_{ij}(-v_ip_ip_j)\}dx \leq 0.$$

If the integrand on the left side is positive definite, then $p_i \equiv 0, i = 1, ..., N$, which means the uniqueness.

But,

$$b_{1j}(-v_1p_1p_j) \leq b_{1j}(-v_1)[\frac{(p_1)^2}{2} + \frac{(p_j)^2}{2}]$$
$$b_{ij}(-v_ip_ip_j) \leq b_{ij}(-v_i)[\frac{(p_i)^2}{2} + \frac{(p_j)^2}{2}]$$

for $j = 2, ..., N, j \neq i$, and so, the result follows if the condition is satisfied by the solution estimates in the Lemma 5.1. □

7. Uniqueness with perturbation

We consider the model

$$\begin{cases} \Delta u_i + u_i(a_i + \sum_{j=1}^N b_{ij}u_j) = 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{7.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $a_i > 0, i = 1, ..., N$ are reproduction rates and b_{ij} are self-limitation and competition rates such that

$$b_{ij} \begin{cases} < 0, i = 1, j = 1, ..., N, \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N. \end{cases}$$

The following theorem is our main result about the perturbation of uniqueness.

Theorem 7.1. *Suppose*
(A) $b_{11}b_{ii} + b_{1i}b_{i1} > 0, i = 2, ..., N,$
(B) $a_1 + \sum_{j=2,j\neq i}^{N} b_{1j}M_j + b_{1i}[\frac{a_1 b_{i1} - a_i b_{11}}{b_{11}b_{ii}}] > \lambda_1, a_1 + \sum_{j=1,j\neq i}^{N} b_{1j}M_j > \lambda_1,$
$a_i + \sum_{j=2}^{N} b_{ij}M_j > \lambda_1, i = 2, ..., N,$
(C) (7.1) has a unique coexistence state $(u_1, ..., u_N)$*,*
(D) the Frechet derivative of (7.1) at $(u_1, ..., u_N)$ *is invertible.*
Then there is a neighborhood V *of* $(a_1, ..., a_N)$ *in* R^N *such that if* $(\bar{a_1}, ..., \bar{a_N}) \in V$*, then (7.1) with* $(\bar{a_1}, ..., \bar{a_N})$ *has a unique positive solution.*

Biologically, the first two conditions in Theorem 7.1 indicates that the rates of reproduction are relatively large, and the birth rate of the prey must be larger than that of predator. Similarly, the fourth condition, which requires the invertibility of the Frechet derivative, signifies that the rates of self-limitation are relatively larger than the rates of competition. When these conditions are fulfilled, the conclusion of our theorem asserts that small perturbations of the rates do not affect the existence and uniqueness of the positive steady state. That is, the two species implied can continue to coexist even if the factors determining the population densities vary slightly.

Now, at first glance, Theorem 7.1 may appear to be a consequence of the Implicit Function Theorem. However, the Implicit Function Theorem only guarantees local uniqueness. In contrast, our result in Theorem 7.1 guarantees global uniqueness. The techniques we will use in the proof of Theorem 7.1 include the Implicit Function Theorem and a priori estimates on solutions of (7.1).

Proof. Since the Frechet derivative of (7.1) at $(u_1, ..., u_N)$ is invertible, by the Implicit Function Theorem, there is a neighborhood V of $(a_1, ..., a_N)$ in R^N and a neighborhood W of $(u_1, ..., u_N)$ in $[C_0^{2,\alpha}(\bar{\Omega})]^N$ such that for all $(\bar{a_1}, ..., \bar{a_N}) \in V$, there is a unique positive solution $(\bar{u_1}, ..., \bar{u_N}) \in W$ of (7.1) with $(\bar{a_1}, ..., \bar{a_N})$. Thus, the local uniqueness of the solution is guaranteed.

To prove global uniqueness, suppose that the conclusion of Theorem 7.1 is false. Then, there are sequences $(a_{1n}, a_{2n}, ..., a_{Nn}, u_{1n}, u_{2n}, ..., u_{Nn})$ and $(a_{1n}, a_{2n}, ..., a_{Nn}, u_{1n}^*, u_{2n}^*, ..., u_{Nn}^*)$ in $V \times [C_0^{2,\alpha}(\bar{\Omega})]^N$ such that $(u_{1n}, ..., u_{Nn})$ and $(u_{1n}^*, ..., u_{Nn}^*)$ are positive solutions of (7.1) with $(a_{1n}, ..., a_{Nn})$, $(u_{1n}, ..., u_{Nn}) \neq (u_{1n}^*, ..., u_{Nn}^*)$ and $(a_{1n}, ..., a_{Nn}) \to (a_1, ..., a_N)$. By Schauder's estimate in elliptic theory and the solution estimate in the Lemma 5.1, there are uniformly convergent subsequences of $\{u_{1n}\}, ..., \{u_{Nn}\}$, which again will be denoted by $\{u_{1n}\}, ..., \{u_{Nn}\}$.

Thus, let

$$(u_{1n}, ..., u_{Nn}) \to (\bar{u_1}, ..., \bar{u_N}),$$
$$(u_{1n}^*, ..., u_{Nn}^*) \to (u_1^*, ..., u_N^*).$$

Then $(\bar{u_1}, ..., \bar{u_N}), (u_1^*, ..., u_N^*) \in (C^{2,\alpha})^N$ are also solutions to (7.1) with $(a_1, ..., a_N)$. We claim that $\bar{u_1} > 0, ..., \bar{u_N} > 0, u_1^* > 0, ..., u_N^* > 0$. By the Maximum Principles, it suffices to claim $\bar{u_1}, ..., \bar{u_N}, u_1^*, ..., u_N^*$ are not identically zero.

Suppose that it is not true. Then by the Maximum Principles again, either one of the followings will hold:
(1) $\bar{u_1} \equiv 0$ and $\bar{u_i} \equiv 0$ for all $i = 2, ..., N$.
(2) $\bar{u_1} \equiv 0$ and $\bar{u_i} > 0$ for some $i = 2, ..., N$.
(3) $\bar{u_1} > 0$ and $\bar{u_i} \equiv 0$ for some $i = 2, ..., N$.
First, suppose $\bar{u_1} \equiv 0$.

Let $\tilde{u_{1n}} = \frac{u_{1n}}{\|u_{1n}\|_\infty}$ and $\tilde{u_{in}} = u_{in}$ for $i = 2, ..., N$.
Then

$$\Delta\tilde{u_{1n}} + \tilde{u_{1n}}(a_{1n} + b_{11}u_{1n} + \sum_{j=2}^N b_{1j}\tilde{u_{jn}}) = 0,$$
$$\Delta\tilde{u_{in}} + \tilde{u_{in}}(a_{in} + b_{i1}u_{1n} + \sum_{j=2}^N b_{ij}\tilde{u_{jn}}) = 0, i = 2, ..., N.$$

By the elliptic theory again, there is $\tilde{u_1}$ such that $\tilde{u_{1n}} \to \tilde{u_1}$, and so,

$$\Delta\tilde{u_1} + \tilde{u_1}(a_1 + \sum_{j=2}^N b_{1j}\bar{u_j}) = 0,$$
$$\Delta\bar{u_i} + \bar{u_i}(a_i + \sum_{j=2}^N b_{ij}\bar{u_j}) = 0, i = 2, ..., N.$$

Hence, $\lambda_1(-a_1 - \sum_{j=2}^N b_{1j}\bar{u_j}) = 0$.

If $\bar{u_i} \equiv 0$ for all $i = 2, ..., N$, then $\lambda_1 - a_1 = \lambda_1(-a_1) = 0$, which is a contradiction to our assumption. If $\bar{u_i} > 0$ for some $i = 2, ..., N$, then by the monotonicity,

$$\begin{aligned} &\lambda_1 - (a_i + \sum_{j=2}^N b_{ij}M_j) \\ &= \lambda_1(-a_i - \sum_{j=2}^N b_{ij}M_j) \\ &\geq \lambda_1(-a_i - \sum_{j=2}^N b_{ij}\bar{u_j}) \\ &= 0, \end{aligned}$$

which is a contradiction.

Suppose $\bar{u_1} > 0$ and $\bar{u_i} \equiv 0$ for some $i = 2, ..., N$. Then

$$\Delta u_{1n} + u_{1n}(a_{1n} + \sum_{j=1}^N b_{1j}u_{jn}) = 0.$$

So,

$$\Delta\bar{u_1} + \bar{u_1}(a_1 + \sum_{j=1, j\neq i}^N b_{1j}\bar{u_j}) = 0.$$

Therefore,

$$\begin{aligned} &\lambda_1 - (a_1 + \sum_{j=1, j\neq i}^N b_{1j}M_j) \\ &= \lambda_1(-a_1 - \sum_{j=1, j\neq i}^N b_{1j}M_j) \\ &\geq \lambda_1(-a_1 - \sum_{j=1, j\neq i}^N b_{1j}\bar{u_j}) \\ &= 0, \end{aligned}$$

which is a contradiction.

Consequently, $(\bar{u_1}, ..., \bar{u_N})$ and $(u_1^*, ..., u_N^*)$ are positive solutions to (7.1) with $(g_1, ..., g_N)$, and so $(\bar{u_1}, ..., \bar{u_N}) = (u_1^*, ..., u_N^*) = (u_1, ..., u_N)$ by the uniqueness condition. But, this is a contradiction to the Implicit Function Theorem, since $(u_{1n}, ..., u_{Nn}) \neq (u_{1n}^*, ..., u_{Nn}^*)$. □

In biological terms, the proof of our theorem indicates that if one of two species living in the same domain becomes extinct, that is, if one species is excluded by the other, then the reproduction rates of both must be small. In other words, the region condition of reproduction rates (A) is reasonable.

Now, the condition (C) in Theorem 7.1 requiring the invertibility of the Frechet derivative is too artificial to have any direct biological implications. We therefore turn our attention to more applicable conditions that will guarantee the invertibility of the Frechet derivative. We then obtain the following relationship:

$$\begin{cases} \Delta u_i + u_i(a_i + \sum_{j=1}^N b_{ij}u_j) = 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{7.2}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $a_i > 0, i = 1, ..., N$ are reproduction rates and b_{ij} are self-limitation and competition rates such that

$$b_{ij} \begin{cases} < 0, i = 1, j = 1, ..., N, \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N. \end{cases}$$

Lemma 7.1. *Suppose* $(u_1, ..., u_N)$ *is a positive solution to (7.2). If*

$$\begin{aligned} &2b_{11} + \textstyle\sum_{i=2}^N b_{i1}\frac{u_i}{u_1} < \sum_{j=2}^N (b_{1j} + b_{j1}\frac{u_j}{u_1}), \\ &2b_{ii} + b_{i1} < \textstyle\sum_{j=2, j\neq i}^N (b_{ij} + b_{ji}\frac{u_j}{u_i}), i = 2, ..., N, \end{aligned}$$

then the Frechet derivative of (7.2) at $(u_1, ..., u_N)$ *is invertible.*

Proof. The Frechet derivative of (7.2) at $(u_1, ..., u_N)$ is $A = (A_{ij})$, where

$$A_{ij} = \begin{pmatrix} -\Delta - (a_i + \sum_{j=1}^N b_{ij}u_j) - b_{ii}u_i, & i = j \\ -b_{ij}u_i, & i \neq j \end{pmatrix}$$

for $i, j = 1, ..., N$. We need to show that $N(A) = \{0\}$ by the Fredholm Alternative, where $N(A)$ is the null space of A. In fact, from the equations

$$\int_\Omega |\nabla\phi_1|^2 - (a_1 + \sum_{j=1}^N b_{1j}u_j + b_{11}u_1)\phi_1^2 - (b_{12}\phi_2 + \cdots + b_{1N}\phi_N)u_1\phi_1 dx = 0,$$

$$\int_\Omega |\nabla\phi_2|^2 - (a_2 + \sum_{j=1}^N b_{2j}u_j + b_{22}u_2)\phi_2^2 - (b_{21}\phi_1 + b_{23}\phi_3 + \cdots + b_{2N}\phi_N)u_2\phi_2 dx = 0,$$

$$\vdots$$

$$\int_\Omega |\nabla\phi_N|^2 - (a_N + \sum_{j=1}^N b_{Nj}u_j + b_{NN}u_N)\phi_N^2 - (b_{N1}\phi_1 + \cdots + b_{N(N-1)}\phi_{N-1})u_N\phi_N dx$$
$$= 0,$$

since $\lambda_1(-a_i - \sum_{j=1}^N b_{ij}u_j) = 0$ for $i = 1, ..., N$, we see that

$$\int_\Omega |\nabla\phi_i|^2 - (a_i + \sum_{j=1}^N b_{ij}u_j)\phi_i^2 dx \geq 0, i = 1, ..., N.$$

Hence,

$$\int_\Omega -b_{11}u_1\phi_1^2 - (b_{12}\phi_2 + \cdots + b_{1N}\phi_N)u_1\phi_1 dx \leq 0,$$

$$\int_\Omega -b_{22}u_2\phi_2^2 - (b_{21}\phi_1 + b_{23}\phi_3 + \cdots + b_{2N}\phi_N)u_2\phi_2 dx \le 0,$$

$$\vdots$$

$$\int_\Omega -b_{NN}u_N\phi_N^2 - (b_{N1}\phi_1 + \cdots + b_{N(N-1)}\phi_{N-1})u_N\phi_N dx \le 0.$$

Therefore,

$$\int_\Omega -\sum_{i=1}^N b_{ii}u_i\phi_i^2 - \sum_{i=1}^N u_i\phi_i \sum_{j=1,j\ne i}^N b_{ij}\phi_j dx \le 0.$$

It implies that

$$\int_\Omega \sum_{i=1}^N (-b_{ii}u_i\phi_i^2 - \sum_{j=1,j\ne i}^N b_{ij}u_i\phi_j\phi_i)dx \le 0.$$

Hence,

$$\begin{aligned}&\textstyle\int_\Omega -b_{11}u_1(\phi_1)^2 - \sum_{j=2}^N b_{1j}u_1\phi_j\phi_1 + \sum_{i=2}^N(-b_{ii}u_i(\phi_i)^2 - b_{i1}u_i\phi_1\phi_i \\ &\textstyle- \sum_{j=2,j\ne i}^N b_{ij}u_i\phi_j\phi_i)dx \le 0.\end{aligned}$$

But, since $-b_{i1}u_i\phi_1\phi_i \ge -b_{i1}u_i(\frac{(\phi_1)^2}{2} + \frac{(\phi_i)^2}{2})$ for all $i = 2, ..., N$,

$$\begin{aligned}&\textstyle\int_\Omega -b_{11}u_1(\phi_1)^2 - \sum_{j=2}^N b_{1j}u_1\phi_j\phi_1 + \sum_{i=2}^N[-b_{ii}u_i(\phi_i)^2 \\ &\textstyle-b_{i1}u_i(\frac{(\phi_1)^2}{2} + \frac{(\phi_i)^2}{2}) - \sum_{j=2,j\ne i}^N b_{ij}u_i\phi_j\phi_i]dx \le 0.\end{aligned}$$

Hence,

$$\begin{aligned}&\textstyle\int_\Omega[-b_{11}u_1 - \sum_{i=2}^N \frac{b_{i1}}{2}u_i](\phi_1)^2 - \sum_{j=2}^N b_{1j}u_1\phi_j\phi_1 + \sum_{i=2}^N[(-b_{ii}u_i - \frac{b_{i1}}{2})(\phi_i)^2 \\ &\textstyle- \sum_{j=2,j\ne i}^N b_{ij}u_i\phi_j\phi_i]dx \le 0.\end{aligned}$$

Since

$$\begin{aligned}&-b_{1j}u_1\phi_j\phi_1 \le -b_{1j}u_1[\tfrac{(\phi_j)^2}{2} + \tfrac{(\phi_1)^2}{2}], j = 2, ..., N, \\ &-b_{ij}u_i\phi_j\phi_i \le -b_{ij}u_i[\tfrac{(\phi_j)^2}{2} + \tfrac{(\phi_i)^2}{2}], i, j = 2, ..., N, i \ne j,\end{aligned}$$

if

$$\begin{aligned}&\textstyle-b_{11}u_1 - \frac{1}{2}\sum_{i=2}^N b_{i1}u_i > -\frac{1}{2}\sum_{j=2}^N(b_{1j}u_1 + b_{j1}u_j], \\ &\textstyle-b_{ii}u_i - \frac{b_{i1}}{2}u_i > -\frac{1}{2}\sum_{j=2,j\ne i}^N(b_{ij}u_i + b_{ji}u_j), i = 2, ..., N,\end{aligned}$$

then the integrand in above inequality is positive definite, which means $(\phi_1, ..., \phi_N)$ is trivial. But, it holds if the conditions are satisfied. □

Combining Lemma 5.1, Theorem 6.1, Theorem 7.1, and Lemma 7.1, we obtain the following corollary.

Corollary 7.1. *If*

(A) $b_{11}b_{ii} + (N-1)b_{1i}b_{i1} > 0$ *for* $i = 2, ..., N$,

(B) $a_1 > \frac{b_{11}b_{ii}(\lambda_1 + \frac{(N-1)b_{1i}a_i}{b_{ii}})}{b_{11}b_{ii} + (N-1)b_{1i}b_{i1}}$, $a_i > \lambda_1 - (N-2)\inf_{j=2,...,N,j\ne i}\{b_{ij}\bar{u_j}\}$,

(C) $a_1 + \sum_{j=2,j\neq i}^{N} b_{1j}M_j + b_{1i}(\frac{a_1 b_{i1} - a_i b_{11}}{b_{11}b_{ii}}) > \lambda_1,\ a_1 + \sum_{j=1,j\neq i}^{N} b_{1j}M_j > \lambda_1,$ *and* $a_i + \sum_{j=2,j\neq i}^{N} b_{ij}M_j > \lambda_1, i = 2, ..., N,$ *and*

(D) $2b_{11} + \sum_{i=2}^{N} b_{i1} \sup \frac{-\frac{1}{b_{ii}}\theta_{a_i - \frac{a_1 b_{i1}}{b_{11}}}}{-\frac{1}{b_{11}}\theta_{a_1 + \sum_{j=2,j\neq i}^{N} b_{1j}M_j + b_{1i}\frac{a_1 b_{i1} - a_i b_{11}}{b_{11}b_{ii}}}} < \sum_{j=2}^{N}(b_{1j}$

$+ b_{j1} \inf \frac{-\frac{1}{b_{jj}}\theta_{a_j + \sum_{k=2,k\neq j}^{N} b_{jk}M_k}}{-\frac{1}{b_{11}}\theta_{a_1}}),$ *and*

$2b_{ii} + b_{i1} < \sum_{j=2,j\neq i}^{N}(b_{ij} + b_{ji} \sup \frac{-\frac{1}{b_{jj}}\theta_{a_j - \frac{a_1 b_{j1}}{b_{11}}}}{-\frac{1}{b_{ii}}\theta_{a_i + \sum_{k=2,k\neq i}^{N} b_{ik}M_k}})$ *for* $i = 2, ..., N,$

then there is a neighborhood V *of* $(a_1, ..., a_N)$ *such that if* $(\bar{a_1}, ..., \bar{a_N}) \in V$*, then (7.1) with* $(\bar{a_1}, ..., \bar{a_N})$ *has a unique positive solution.*

In biological terms, the result obtained in Corollary 7.1 confirms that under certain conditions, two species who relax ecologically can continue to coexist at fixed rates. The requirements given in (A) and (B) simply state that each species must interact strongly with itself and weakly with the other species.

8. Uniqueness with perturbation of region

We consider the model

$$\begin{cases} \Delta u_i + u_i(a_i + \sum_{j=1}^{N} b_{ij}u_j) = 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{8.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $a_i > 0, i = 1, ..., N$ are reproduction rates and b_{ij} are self-limitation and competition rates such that

$$b_{ij} \begin{cases} < 0, i = 1, j = 1, ..., N, \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N. \end{cases}$$

The following Theorem is the main result.

Theorem 8.1. *Suppose that* $\Gamma \subseteq R^N$ *is a closed, bounded, convex region such that*
(A) $b_{11}b_{ii} + b_{1i}b_{i1} > 0, i = 2, ..., N,$
(B) for all $(a_1, ..., a_N) \in \Gamma,$
$a_1 + \sum_{j=2,j\neq i}^{N} b_{1j}M_j + b_{1i}(\frac{a_1 b_{i1} - a_i b_{11}}{b_{11}b_{ii}}] > \lambda_1, a_1 + \sum_{j=1,j\neq i}^{N} b_{1j}M_j > \lambda_1,$
$a_i + \sum_{j=2,j\neq i}^{N} b_{ij}M_j > \lambda_1, i = 2, ..., N,$
C) for all $(a_1, ..., a_N) \in \partial_L\Gamma$*, (8.1) has a unique positive solution, where* $\partial_L\Gamma = \{(\lambda_{a_2,...,a_N}, a_2, ..., a_N) \in \Gamma|$ *for any fixed* $a_2, ..., a_N,$
$\lambda_{a_2,...,a_N} = \inf\{|a_1||(a_1, a_2, ..., a_N) \in \Gamma\}.\},$
(D) for all $(a_1, ..., a_N) \in \Gamma$*, the Frechet derivative of (8.1) at every positive solution* $(u_1, ..., u_N)$ *is invertible.*

Then for all $(a_1, ..., a_N) \in \Gamma$*, (8.1) has a unique positive solution. Furthermore, there is an open set* W *in* B *such that* $\Gamma \subseteq W$ *and for every* $(g_1, ..., g_N) \in W$*, (8.1) has a unique positive solution.*

Theorem 8.1 goes even further than Theorem 7.1 which states uniqueness in the whole region whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed $a_2, ..., a_N$, consider $(a_1, a_2, ..., a_N) \in \partial_L \Gamma$ and $(\bar{a_1}, a_2, ..., a_N) \in \Gamma$. We need to show that for all $0 \le t \le 1$, (8.1) with $(1-t)(a_1, ..., a_N) + t(\bar{a_1}, a_2, ..., a_N)$ has a unique positive solution. Since (8.1) with $(a_1, ..., a_N)$ has a unique positive solution $(u_1, ..., u_N)$ and the Frechet derivative of (8.1) at $(u_1, ..., u_N)$ is invertible, Theorem 7.1 implies that there is an open neighborhood V of $(a_1, ..., a_N)$ in such that if $(a_{10}, ..., a_{N0}) \in V$, then (8.1) with $(a_{10}, ..., a_{N0})$ has a unique positive solution.

Let $\lambda_s = \sup\{0 \le \lambda \le 1| \quad (8.1) \text{ with } \quad (1-t)(a_1, ..., a_N) + t(\bar{a_1}, a_2, ..., a_N)$ has a unique coexistence state for $\quad 0 \le t \le \lambda\}$. We need to show that $\lambda_s = 1$. Suppose $\lambda_s < 1$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and there is a sequence $(u_{1n}, ..., u_{Nn})$ of the unique positive solutions of (8.1) with $(1-\lambda_n)(a_1, ..., a_N) + \lambda_n(\bar{a_1}, a_2, ..., a_N)$. Then by elliptic theory, there is $(u_{10}..., , u_{N0})$ such that $(u_{1n}, ..., u_{Nn})$ converges to $(u_{10}, ..., u_{N0})$ uniformly and $(u_{10}, ..., u_{N0})$ is a solution of (8.1) with $(1-\lambda_s)(a_1, ..., a_N) + \lambda_s(\bar{a_1}, a_2, ..., a_N)$. But, by the same proof as in Section 7, $u_{10} > 0, ..., u_{N0} > 0$.

We claim that (8.1) has a unique coexistence state with $(1-\lambda_s)(a_1, ..., a_N) + \lambda_s(\bar{a_1}, a_2, ..., a_N)$. In fact, if not, assume that $(\bar{u}_{10}, ..., \bar{u}_{N0}) \ne (u_{10}, ..., u_{N0})$ is another coexistence state. By the Implicit Function Theorem, there exists $0 < \tilde{a} < \lambda_s$ and very close to λ_s such that (8.1) with $(1-\tilde{a})(a_1, ..., a_N) + \tilde{a}(\bar{a_1}, a_2, ..., a_N)$ has a coexistence state very close to $(\bar{u}_{10}, ..., \bar{u}_{N0})$, which means that (8.1) with $(1-\tilde{a})(a_1, ..., a_N) + \tilde{a}(\bar{a_1}, a_2, ..., a_N)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (8.1) with $(1-\lambda_s)(a_1, ..., a_N) + \lambda_s(\bar{a_1}, a_2, ..., a_N)$ has a unique coexistence state and the Frechet derivative is invertible, Theorem 7.1 implies that λ_s can not be as defined. Therefore, for each $(a_1, .., a_N) \in \Gamma$, (8.1) with $(a_1, ..., a_N)$ has a unique coexistence state $(u_1, ..., u_N)$. Furthermore, by the assumption, for each $(a_1, ..., a_N) \in \Gamma$, the Frechet derivative of (8.1) with $(a_1, ..., a_N)$ at the unique solution $(u_1, ..., u_N)$ is invertible. Hence, Theorem 7.1 concluded that for each $(a_1, ..., a_N) \in \Gamma$, there is an open neighborhood $V_{(a_1,...,a_N)}$ of $(a_1, ..., a_N)$ such that if $(\bar{a_1}, ..., \bar{a_N}) \in V_{(a_1,...,a_N)}$, then (8.1) with $(\bar{a_1}, ..., \bar{a_N})$ has a unique coexistence state. Let $W = \bigcup_{(a_1,...,a_N)\in\Gamma} V_{(a_1,...,a_N)}$. Then W is an open set in B such that $\Gamma \subseteq W$ and for each $(\bar{a_1}, ..., \bar{a_N}) \in W$, (8.1) with $(\bar{a_1}, ..., \bar{a_N})$ has a unique coexistence state. □

Apparently, Theorem 8.1 generalizes Theorem 7.1.

9. Conclusions

The Theorem 4.1 indicates that if the species of animals have strong enough birth capacities, then they may peacefully coexist forever. Furthermore, the Theorem 6.1 implies that if their self-limitations are stronger than competitions, in other words, they interact stronger among themselves than with others, then their coexistence pattern is unique. We also concluded in Theorems 4.2 and 4.3 that either one of them may be extinct if they don't have strong birth rates. Our investigation of the effects of perturbations on the general predator-prey model resulted in the development and proof of Theorem 7.1, Lemma 7.1, and Corollary 7.1 as detailed above. The three together assert that given the existence of a unique solution $(u_1, ..., u_N)$

to the system (7.1), perturbations of the birth rates $(a_1, ..., a_N)$, within a specified neighborhood, will maintain the existence and uniqueness of the positive steady state. Indeed, our results specifically outline conditions sufficient to maintain the positive, steady state solution when the general predator-prey model is perturbed within some region.

Applying this mathematical result to real world situations, our results establish that the species residing in the same environment can vary their interactions, within certain bounds, and continue to survive together indefinitely at unique densities. The conditions necessary for coexistence, as described in the theorem, simply require that members of each species interact strongly with themselves and weakly with members of the other species.

The research presented in this paper has a number of strengths, which confirm both the validity and the applicability of the project. First, the mathematical conditions required in Corollary 7.1 are identical to those required in Theorem 6.1. However, in the Theorem 6.1, we used these conditions to prove the existence and uniqueness of the positive steady state solution for the general predator-prey model. In contrast, the Corollary 7.1 employs the same conditions to establish that the existence and uniqueness of this solution is maintained when the model is perturbed within some neighborhood. Thus, our findings extend and improve established mathematical theory.

Secondly, perturbations of the general model render its implications more applicable both mathematically and biologically. Because our theorem extends the steady state to any value within some neighborhood of $(a_1, ..., a_N)$, results for the general model pertain to a far wider variety of values. Biologically, perturbations extend the model's description to species affected by factors that vary slightly yet erratically. Thus, the description of competitive interactions given by the model becomes a closer approximation of real-world population dynamics.

While our research therefore represents a progression in the field, the results obtained have an important limitation. Theorem 7.1, Lemma 7.1, and Corollary 7.1 establish that a region of perturbation exists within which the coexistence state is maintained for the general predator-prey model. However, the exact extent of that region remains unknown. Therefore, the results presented in this paper may serve as a platform for research of the question given above. Mathematicians should now attempt to establish the exact extent of the perturbation region in which coexistence is maintained for the general model. Such information would prove very useful not only mathematically but also biologically. Specifically, knowledge of the extent of the region would imply exactly how far the species can relax and yet continue to coexist. Thus, the results achieved through our research will enable the field to continue the development of theory on predator-prey interaction of populations.

In the future, we also want to develop population models to generalize mathematical results about other models related to COVID-19.(see [20], [21], [22], [23]).

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