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Solvability for a Class of Logarithmic Type Initial Conditions of Hadamard Fractional Differential System on an Infinite Interval*

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Abstract This paper investigates a class of Hadamard fractional differential system involving logarithmic type initial conditions on an infinite interval. Based on some obtained properties of the Green's functions, Schauder fixed point theorem and Banach contraction mapping principle, we establish the existence and uniqueness results for the system. Finally, we illustrate examples which show our main results.

Keywords Solvability, Hadamard fractional differential system, logarithmic type initial conditions, infinite interval

MSC(2010) 26A33, 34A37, 34B15, 34B18.

1. Introduction

We examine a system comprising nonlinear Hadamard fractional differential equations (HFDEs for short)

$$\begin{cases} D^{p}_{1^+}x(t)+f(t,x(t),D^{q-1}_{1^+}y(t))=0, t\in(1,\infty),\\ D^{q}_{1^+}y(t)+g(t,x(t),D^{p-1}_{1^+}y(t))=0, t\in(1,\infty),\end{cases} \tag{1.1}$$

supplemented with some logarithmic type initial conditions

$$\begin{cases} \lim_{t\to1}(\log t)^{2-p}x(t)=\lim_{t\to\infty}D^{p-1}_{1^+}x(t)=\sum_{i=1}^{m}a_ix(\eta_i),\\ \lim_{t\to1}(\log t)^{2-q}y(t)=\lim_{t\to\infty}D^{q-1}_{1^+}y(t)=\sum_{j=1}^{n}b_jy(\zeta_j),\end{cases} \tag{1.2}$$

where $D^{v}_{1^+}$ is the Hadamard type fractional derivative of order $v\in\{p,q\}, p,q\in(1,2], f,g\in C([1,\infty)\times\mathbf{R}\times\mathbf{R},\mathbf{R}), 1<\eta_1<\eta_2<\cdots<\eta_m<\infty, 1<\zeta_1<\zeta_2<\cdots<\zeta_n<\infty, a_i(i=1,2,\ldots,m), b_j(j=1,2,\ldots,n)$ denote positive real constants with

$$1-\sum_{i=1}^{m}a_i(\frac{(\log\eta_i)^{p-1}}{\Gamma(p)}+(\log\eta_i)^{p-2})>0,$$

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$$1-\sum_{j=1}^{n}b_j(\frac{(\log\zeta_j)^{q-1}}{\Gamma(q)}+(\log\zeta_j)^{q-2})>0.$$

For variables in infinite intervals, we have to consider different function spaces and infinite integrals. Therefore, we need the following assumptions:
(A_1) $f:[1,\infty)\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, satisfies the a-Caratheodory conditions, that is
(i) $t\to f(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-p}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is measurable on $[1,\infty)$ for every $(x,y)\in\mathbf{R}^2$;
(ii) $(x,y)\to f(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-p}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is continuous on $\mathbf{R}^2$ for all $t\in[1,\infty)$;
(iii) for any $r>0$, there exists a function $\varphi_r(t)\geq 0$ with $\int_1^\infty\varphi_r(t)\frac{dt}{t}<\infty$, such that

$$f(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-p}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)\leq\varphi_r(t)\text{ for }t\in[1,\infty),|x|,|y|\leq r.$$

(A_2) $g:[1,\infty)\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, satisfies the a-Caratheodory conditions, that is
(i) $t\to g(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-q}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is measurable on $[1,\infty)$ for every $(x,y)\in\mathbf{R}^2$;
(ii) $(x,y)\to g(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-q}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is continuous on $\mathbf{R}^2$ for all $t\in[1,\infty)$;
(iii) for any $r>0$, there exists a function $\phi_r(t)\geq 0$ with $\int_1^\infty\phi_r(t)\frac{dt}{t}<\infty$, such that

$$g(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-q}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)\leq\phi_r(t)\text{ for }t\in[1,\infty),|x|,|y|\leq r.$$

(A_3) $f(t,0,0),g(t,0,0)\not\equiv 0$ on any subinterval of $[1,\infty)$;

Fractional calculus is an important branch of mathematics, which is used to study the differentiation and integration of any real order or complex order, and can be used to solve many problems which are difficult to be solved by integral calculus. In the past few decades, fractional differential equations (FDEs for short) have been widely used in physics, mechanics, chemistry, economics, biomedicine and other fields, which are mainly used to describe the behavior of complex systems, describe the properties of complex media mechanics, describe molecular diffusion behavior, establish nonlinear economic models, describe biomedical signals and so on, see [1–4]. So far, several different fractional derivatives have been proposed, for example, the Riemann-Liouville, Caputo, Hadamard, and Caputo-Fabrizio fractional derivatives, see [5–13, 24]. Different derivative operators are often used according to their different structures. In this paper, we mainly study FDEs with Hadamard type derivatives. In order to gain insight into the many applications of fractional calculus in different fields, we refer the reader to [14–17, 25–41].

In [22], Nyamoradi and Ahmad applied the Leggett-Williams fixed point theorem and the concept of iterative positive solutions to prove the existence of at least two or three positive solutions for the following HFDE:

$$\begin{cases}D_{1^+}^{\varsigma}x(t)+\upsilon(t)f(t,x(t))=0,t\in(1,\infty),\\ \lim_{t\to 1}(\log t)^{2-\varsigma}x(t)=\lim_{t\to\infty}D_{1^+}^{\varsigma-1}x(t)=\int_1^\infty\tau(s)x(s)\frac{ds}{s},\end{cases}\tag{1.3}$$

where $D_{1^+}^{\varsigma}$ is the Hadamard fractional derivative of order $\varsigma\in(1,2]$, $f:(1,\infty)\times\mathbf{R}\to\mathbf{R}$, $\upsilon:(1,\infty)\to(1,\infty)$ are continuous functions, $0<\int_1^\infty\upsilon(s)\frac{ds}{s}<\infty,\tau\in L^1(1,\infty)$ and

$$\Psi_\varsigma=\int_1^\infty\left(\frac{(\log s)^{\varsigma-1}}{\Gamma(\varsigma)}+(\log s)^{\varsigma-2}\right)\tau(s)\frac{ds}{s}<1.$$

In [23], Tariboon et al. employed the Leggett-Williams fixed point theorem and Guo-Krasnoselskii's fixed point theorem to investigate the existence of positive solutions for the following HFDEs:

$$\begin{cases} D_{1+}^{\alpha}x(t)+u(t)f(t,x(t),y(t))=0, 1<\alpha\le 2, t\in(1,\infty),\\ D_{1+}^{\beta}y(t)+v(t)g(t,x(t),y(t))=0, 1<\beta\le 2, t\in(1,\infty),\\ x(1)=0, D_{1+}^{\alpha-1}x(\infty)=\sum_{i=1}^{m}\rho_i I_{1+}^{p_i}y(\eta),\\ y(1)=0, D_{1+}^{\beta-1}y(\infty)=\sum_{j=1}^{n}\varrho_j I_{1+}^{q_j}x(\zeta),\end{cases} \tag{1.4}$$

where D_{1+}^{r} is the Hadamard type fractional derivative of order $r=\alpha,\beta$; $I_{1+}^{p_i}, I_{1+}^{q_j}$ are the Hadamard type fractional integrals of orders $p_i, q_j\ge 1$, respectively; $\rho_i,\varrho_j>0, i=1,2,\dots m, j=1,2,\dots n$.

In [21], Deren and Cerdik applied the monotone iterative technique to find the existence of positive solutions for the following system of HFDEs with multi-point boundary conditions:

$$\begin{cases} D_{1+}^{p}x(t)+w_1(t)f(t,y(t),D_{1+}^{p-1}y(t))=0, n-1<p\le n, t\in(1,\infty),\\ D_{1+}^{q}y(t)+w_2(t)g(t,x(t),D_{1+}^{p-1}x(t))=0, m-1<q\le m, t\in(1,\infty),\\ x(1)=x'(1)=\cdots=x^{(n-2)}(1)=0, D_{1+}^{p-1}x(\infty)=\Sigma_{i=1}^{k_1}a_iD_{1+}^{r_1}x(n_i),\\ y(1)=y'(1)=\cdots=y^{(m-2)}(1)=0, D_{1+}^{q-1}y(\infty)=\Sigma_{j=1}^{k_2}b_jD_{1+}^{r_2}y(m_j),\end{cases} \tag{1.5}$$

where D_{1+}^{v} is Hadamard type fractional derivative of order $v\in\{p,q,r_1,r_2\}, r_1\in[0,p-1], r_2\in[0,q-1], a_i\ge 0(i=1,2,\dots k_1), b_j\ge 0(j=1,2,\dots k_2), 1<n_1<n_2<\cdots<n_{k_1}<\infty$ and $1<m_1<m_2<\cdots<m_{k_2}<\infty$.

In this paper, we aim to obtain the existence and uniqueness of solutions for the system (1.1)-(1.2). To date, there are still few studies considering the systems of HFDEs on infinite intervals. The Schauder fixed point theorem and Banach contraction mapping principle are applied to derive the desired results. Our results are new in the given configuration and open new avenues for further investigation of (1.1)-(1.2).

The structure of the paper is outlined as follows. In Section 2, we list some useful preliminaries and the key lemmas that are used in subsequent parts, and then we obtain some useful properties of Green's functions. In Section 3, we set out the main conclusion: the existence and uniqueness results of solutions. Section 4 showcases a selection of examples that serve to illustrate our results.

2. Preliminaries

In the section, we recall some basic concepts, notations and related lemmas.

Definition 2.1. ([18, 19]) For a function c, the Hadamard fractional derivative of order v is

$$D_{1+}^{v}c(t)=\frac{1}{\Gamma(n-v)}(t\frac{d}{dt})^{n}\int_{1}^{t}(\log\frac{t}{s})^{n-v-1}c(s)\frac{ds}{s}, n-1<v<n,$$

where $[v]$ denotes the integer part of the real number $v, n=[v]+1$ and $\log(.)=\log_e(.)$.

Definition 2.2. ([18, 19]) For a function c, the Hadamard fractional integral of order v is

$$I_{1^+}^{v}c(t)=\frac{1}{\Gamma(v)}\int_1^t(\log\frac{t}{s})^{v-1}c(s)\frac{ds}{s}, v>0,$$

assuming the integral exists.

Lemma 2.1 ([19]). *If* $\alpha,\beta>0$, *then*

$$I_{1^+}^{\alpha}(\log t)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\log t)^{\beta+\alpha-1},$$

$$D_{1^+}^{\alpha}(\log t)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\log t)^{\beta-\alpha-1}.$$

In particular, $D_{1^+}^{\alpha}(\log t)^{\alpha-j}=0, j=1,2,\dots,[\alpha]+1.$

Lemma 2.2 ([19]). *Let* $\xi>0$ *and* $x\in C[1,\infty)\cap L^1[1,\infty)$. *Then the solution of HFDE* $D_{1^+}^{\xi}x(t)=0$ *is written as*

$$x(t)=\sum_{i=1}^{n}c_i(\log t)^{\xi-i},$$

and the following formula satisfies

$$I_{1^+}^{\xi}D_{1^+}^{\xi}x(t)=x(t)+\sum_{i=1}^{n}c_i(\log t)^{\xi-i},$$

where $c_i\in\boldsymbol{R}, i=1,2,\dots,n, n=[\xi]+1.$

Next, we introduce some spaces related to our work as follows:

$$\begin{aligned}
X_1=&\left\{x\in C[1,\infty):\sup_{t\in[1,\infty)}\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}|x(t)|<\infty,\sigma>-1\right\},\\
X=&\left\{x\in X_1, D_{1^+}^{p-1}x\in C[1,\infty):\sup_{t\in[1,\infty)}\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{p-1}x(t)|<\infty,\sigma>-1\right\},\\
Y_1=&\left\{y\in C[1,\infty):\sup_{t\in[1,\infty)}\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}|y(t)|<\infty,\sigma>-1\right\},\\
Y=&\left\{y\in Y_1, D_{1^+}^{q-1}y\in C[1,\infty):\sup_{t\in[1,\infty)}\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{q-1}y(t)|<\infty,\sigma>-1\right\}.
\end{aligned}$$

For $x\in X$, define the norm by

$$\|x\|_X=\max\left\{\sup_{t\in[1,\infty)}\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}|x(t)|,\ \sup_{t\in[1,\infty)}\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{p-1}x(t)|\right\}.$$

It is easy to show that X is a real Banach space.

For $y\in Y$, define the norm by

$$\|y\|_Y=\max\left\{\sup_{t\in[1,\infty)}\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}|y(t)|,\ \sup_{t\in[1,\infty)}\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{q-1}y(t)|\right\}.$$

Then Y is also a real Banach space. And thus the product space $(X\times Y,\|\cdot\|)$ is a Banach space with norm

$$\|(x,y)\|=\max\{\|x\|_X,\|y\|_Y\}.$$

To discuss (1.1),(1.2), we need to transform it into a fixed point problem, so we first give the following conclusion. Although the method is usual, we give the proof for completion.

Lemma 2.3. *Let $h\in C[1,\infty)$. There exists a constant $M>0$ such that $|h(t)|\le M$, $\int_1^\infty h(s)\frac{ds}{s}<\infty$ and*

$$\triangle_1:=1-\sum_{i=1}^m a_i(\frac{(\log\eta_i)^{p-1}}{\Gamma(p)}+(\log\eta_i)^{p-2})>0,$$

then the solution of the following HFDE with logarithmic type conditions

$$\begin{cases}D_{1^+}^p x(t)+h(t)=0, 1<p\le 2, t\in(1,\infty),\\ \lim_{t\to1}(\log t)^{2-p}x(t)=\lim_{t\to\infty}D_{1^+}^{p-1}x(t)=\sum_{i=1}^m a_ix(\eta_i),\end{cases}\tag{2.1}$$

is given by

$$x(t)=\int_1^\infty G(t,s)h(s)\frac{ds}{s}, t\in[1,\infty),$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s),$$

with

$$G_1(t,s)=\frac{1}{\Gamma(p)}\begin{cases}(\log t)^{p-1}-(\log\frac{t}{s})^{p-1}, & 1\le s\le t<\infty,\\ (\log t)^{p-1}, & 1\le t\le s<\infty,\end{cases}\tag{2.2}$$

$$G_1(\eta_i,s)=\frac{1}{\Gamma(p)}\begin{cases}(\log\eta_i)^{p-1}-(\log\frac{\eta_i}{s})^{p-1}, & 1\le s\le\eta_i<\infty,\\ (\log\eta_i)^{p-1}, & 1\le\eta_i\le s<\infty,\end{cases}\tag{2.3}$$

and

$$G_2(t,s)=\frac{(\frac{(\log t)^{p-1}}{\Gamma(p)}+(\log t)^{p-2})}{1-\Delta_1}\sum_{i=1}^m a_iG_1(\eta_i,s).$$

Proof. Using Lemma 2.2, the solution of (2.1) can be written as

$$x(t)=-\frac{1}{\Gamma(p)}\int_1^t\log(\frac{t}{s})^{p-1}h(s)\frac{ds}{s}+c_1(\log t)^{p-1}+c_2(\log t)^{p-2},$$

from some constants $c_1,c_2\in\mathbf{R}$. When $t\to1$,

$$\left|(\log t)^{2-p}\int_1^t(\log\frac{t}{s})^{p-1}\frac{h(s)}{s}ds\right|\le\ |M(\log t)^{2-p}\int_1^t(\log\frac{t}{s})^{p-1}\frac{1}{s}ds|$$
$$=\frac{M}{p}(\log t)^{2-p}(\log t)^p=\frac{M}{p}(\log t)^2\to0.$$

From $\lim_{t\to1}(\log t)^{2-p}x(t)=\sum_{i=1}^m a_ix(\eta_i)$, we know that

$$c_2=\sum_{i=1}^m a_ix(\eta_i).$$

From Lemma 2.1,

$$D_{1^+}^{p-1}x(t) = -\int_1^t \frac{h(s)}{s}ds + c_1\Gamma(p).$$

Considering the condition $\lim_{t\to\infty} D_{1^+}^{p-1}x(t) = \sum_{i=1}^m a_i x(\eta_i)$, we get

$$c_1 = \frac{1}{\Gamma(p)}\Big(\int_1^\infty h(s)\frac{ds}{s} + \sum_{i=1}^m a_i x(\eta_i)\Big).$$

Hence, we get

$$\begin{aligned} x(t) =& -\frac{1}{\Gamma(p)}\int_1^t \log(\frac{t}{s})^{p-1}h(s)\frac{ds}{s} + (\frac{(\log t)^{p-1}}{\Gamma(p)} \\ &+ (\log t)^{p-2})\sum_{i=1}^m a_i x(\eta_i) + \frac{(\log t)^{p-1}}{\Gamma(p)}\int_1^\infty h(s)\frac{ds}{s}, \end{aligned}$$

and together

$$\begin{aligned} \sum_{i=1}^m a_i x(\eta_i) =& \frac{1}{1-\Delta_1}\Bigg(-\frac{1}{\Gamma(p)}\sum_{i=1}^m a_i\int_1^{\eta_i}\Big(\log\frac{\eta_i}{s}\Big)^{p-1}h(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(p)}\sum_{i=1}^m a_i(\log\eta_i)^{p-1}\int_1^\infty h(s)\frac{ds}{s}\Bigg), \end{aligned}$$

we obtain

$$\begin{aligned} x(t) =& -\frac{1}{\Gamma(p)}\int_1^t \log(\frac{t}{s})^{p-1}h(s)\frac{ds}{s} + \frac{(\log t)^{p-1}}{\Gamma(p)}\int_1^\infty h(s)\frac{ds}{s} \\ &+ \frac{(\frac{(\log t)^{p-1}}{\Gamma(p)} + (\log t)^{p-2})}{1-\Delta_1}\Big(-\frac{1}{\Gamma(p)}\sum_{i=1}^m a_i\int_1^{\eta_i}(\log\frac{\eta_i}{s})^{p-1}h(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(p)}\sum_{i=1}^m a_i(\log\eta_i)^{p-1}\int_1^\infty h(s)\frac{ds}{s}\Big) \\ =& \int_1^\infty G(t,s)h(s)\frac{ds}{s}. \end{aligned}$$

The proof is completed. □

Similarly, for $k \in C[1,\infty)$, there exists a constant $M' > 0$ such that $|k(t)| \le M'$, $\int_1^\infty k(s)\frac{ds}{s} < \infty$ and

$$\triangle_2 := 1 - \sum_{j=1}^n b_j(\frac{(\log\zeta_j)^{q-1}}{\Gamma(q)} + (\log\zeta_j)^{q-2}) > 0.$$

Then the solution of the following HFDE with logarithmic type boundary conditions

$$\begin{cases} D_{1^+}^q y(t) + k(t) = 0, 1 < q \le 2, t \in (1,\infty), \\ \lim_{t\to1}(\log t)^{2-q}y(t) = \lim_{t\to\infty} D_{1^+}^{q-1}y(t) = \sum_{i=1}^m b_j y(\zeta_i), \end{cases} \tag{2.4}$$

is given by

$$y(t) = \int_1^\infty H(t,s)k(s)\frac{ds}{s}, t \in [1,\infty),$$

where

$$H(t,s) = H_1(t,s) + H_2(t,s),$$

with

$$H_1(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (\log t)^{q-1} - (\log \frac{t}{s})^{q-1}, & 1 \le s \le t < \infty, \\ (\log t)^{q-1}, & 1 \le t \le s < \infty, \end{cases} \tag{2.5}$$

$$H_1(\zeta_j,s) = \frac{1}{\Gamma(q)} \begin{cases} (\log \zeta_j)^{q-1} - (\log \frac{\zeta_j}{s})^{q-1}, & 1 \le s \le \zeta_j < \infty, \\ (\log \zeta_j)^{q-1}, & 1 \le \zeta_j \le s < \infty, \end{cases} \tag{2.6}$$

and

$$H_2(t,s) = \frac{(\frac{(\log t)^{q-1}}{\Gamma(q)} + (\log t)^{q-2})}{1-\Delta_2} \sum_{j=1}^{n} b_j H_1(\zeta_j, s).$$

Lemma 2.4. *The functions $G(t,s), H(t,s)$ admit the following properties:*
(i) $G(t,s), H(t,s)$ are nonnegative and continuous for $(t,s) \in [1,\infty) \times [1,\infty)$;
(ii)

$$\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G_1(t,s) \le \frac{1}{\Gamma(p)}, \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} H_1(t,s) \le \frac{1}{\Gamma(q)};$$

(iii)

$$\begin{aligned} \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G_2(t,s) &\le \frac{1+\Gamma(p)}{(1-\Delta_1)\Gamma^2(p)} \sum_{i=1}^{m} a_i (\log \eta_i)^{p-1}, \\ \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} H_2(t,s) &\le \frac{1+\Gamma(q)}{(1-\Delta_2)\Gamma^2(q)} \sum_{j=1}^{n} b_j (\log \zeta_j)^{q-1}; \end{aligned}$$

(iv)

$$\begin{aligned} \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G(t,s) &\le \frac{1}{\Gamma(p)} \left[1 + \frac{1+\Gamma(p)}{(1-\Delta_1)\Gamma(p)} \sum_{i=1}^{m} a_i (\log \eta_i)^{p-1}\right] =: W_1, \\ \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} H(t,s) &\le \frac{1}{\Gamma(q)} \left[1 + \frac{1+\Gamma(q)}{(1-\Delta_2)\Gamma(q)} \sum_{j=1}^{n} b_j (\log \zeta_j)^{q-1}\right] =: W_2; \end{aligned}$$

(v)

$$\begin{aligned} (\log t)^{2-p} G(t,s) &\le \frac{1}{\Gamma(p)} \left(1 + \frac{1}{(1-\Delta_1)\Gamma(p)} \sum_{i=1}^{m} a_i (\log \eta_i)^{p-1}\right) \log t \\ &\quad + \frac{1}{(1-\Delta_1)\Gamma(p)} \sum_{i=1}^{m} a_i (\log \eta_i)^{p-1} \\ &= Q_1 \log t + \frac{1}{(1-\Delta_1)\Gamma(p)} \sum_{i=1}^{m} a_i (\log \eta_i)^{p-1}, \\ (\log t)^{2-q} H(t,s) &\le \frac{1}{\Gamma(q)} \left(1 + \frac{1}{(1-\Delta_2)\Gamma(q)} \sum_{j=1}^{n} b_j (\log \zeta_i)^{q-1}\right) \log t \end{aligned}$$

$$+\frac{1}{(1-\Delta_2)\Gamma(q)}\sum_{j=1}^{n}b_j(\log\zeta_i)^{q-1}$$

$$=Q_2\log t+\frac{1}{(1-\Delta_2)\Gamma(q)}\sum_{j=1}^{n}b_j(\log\zeta_i)^{q-1},$$

where

$$Q_1=\frac{1}{\Gamma(p)}\bigg(1+\frac{1}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}\bigg),$$

$$Q_2=\frac{1}{\Gamma(q)}\bigg(1+\frac{1}{(1-\Delta_2)\Gamma(q)}\sum_{j=1}^{n}b_j(\log\zeta_i)^{q-1}\bigg).$$

Proof. The proofs of (i) and (ii) are straightforward, so we omit it. Next we prove the remaining properties:
(iii)

$$\begin{aligned}&\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G_2(t,s)\\ \le&\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}\frac{(\frac{(\log t)^{p-1}}{\Gamma(p)}+(\log t)^{p-2})}{1-\Delta_1}\sum_{i=1}^{m}a_iG_1(\eta_i,s)\\ \le&\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}\frac{(\frac{(\log t)^{p-1}}{\Gamma(p)}+(\log t)^{p-2})}{1-\Delta_1}\sum_{i=1}^{m}a_i\frac{1}{\Gamma(p)}(\log\eta_i)^{p-1}\\ \le&\frac{1+\Gamma(p)}{(1-\Delta_1)\Gamma^2(p)}\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}.\end{aligned}$$

Similarly,

$$\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}H_2(t,s)\le\frac{1+\Gamma(q)}{(1-\Delta_2)\Gamma^2(q)}\sum_{j=1}^{n}b_j(\log\zeta_j)^{q-1}.$$

(iv)

$$\begin{aligned}\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G(t,s)&=\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G_1(t,s)+\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G_2(t,s)\\ &\le\frac{1}{\Gamma(p)}\bigg[1+\frac{1+\Gamma(p)}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}\bigg].\end{aligned}$$

Further,

$$\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}H(t,s)\le\frac{1}{\Gamma(q)}\bigg[1+\frac{1+\Gamma(q)}{(1-\Delta_2)\Gamma(q)}\sum_{j=1}^{n}b_j(\log\zeta_j)^{q-1}\bigg].$$

$$(v)\ (\log t)^{2-p}G(t,s)\le\frac{1}{\Gamma(p)}\bigg(\log t+\frac{\log t+\Gamma(p)}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}\bigg)$$

$$\leq \frac{1}{\Gamma(p)}\left(1+\frac{1}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m} a_i(\log\eta_i)^{p-1}\right)\log t$$
$$+\frac{1}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m} a_i(\log\eta_i)^{p-1}$$
$$= Q_1\log t+\frac{1}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m} a_i(\log\eta_i)^{p-1}.$$

Similarly,

$$(\log t)^{2-q}H(t,s)\leq Q_2\log t+\frac{1}{(1-\Delta_2)\Gamma(q)}\sum_{j=1}^{n} b_j(\log\zeta_j)^{q-1}.$$

□

Remark 2.1. In accordance with Definition 2.1, Lemma 2.3, we get

$$D_{1^+}^{p-1}x(t)=\int_1^\infty G^*(t,s)h(s)\frac{ds}{s}, D_{1^+}^{q-1}y(t)=\int_1^\infty H^*(t,s)k(s)\frac{ds}{s},$$

in which

$$G^*(t,s)=J(t,s)+\frac{1}{1-\Delta_1}\sum_{i=1}^{m} a_iG_1(\eta_i,s),$$
$$H^*(t,s)=J(t,s)+\frac{1}{1-\Delta_2}\sum_{j=1}^{n} b_jG_2(\zeta_j,s),$$

and

$$J(t,s)=\begin{cases}0, & 1\leq s\leq t<\infty,\\ 1, & 1\leq t\leq s<\infty.\end{cases}$$

Clearly, for $t,s\in[1,\infty), G^*(t,s), H^*(t,s)$ are continuous and $G^*(t,s), H^*(t,s)\geq 0$, with

$$G^*(t,s)\leq 1+\frac{1}{1-\Delta_1}\sum_{i=1}^{m} a_iG_1(\eta_i,s)=:\Pi_1,$$
$$H^*(t,s)\leq 1+\frac{1}{1-\Delta_2}\sum_{j=1}^{n} b_jG_2(\zeta_j,s)=:\Pi_2.$$

3. Main results

Define an operator T on $X\times Y$ by

$$T(x,y)(t)=(T_1(x,y)(t),T_2(x,y)(t)),$$

where

$$T_1(x,y)(t)=\int_1^\infty G(t,s)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s},$$
$$T_2(x,y)(t)=\int_1^\infty H(t,s)g(s,x(s),D_{1^+}^{p-1}y(s))\frac{ds}{s}.$$

We find that the existence of a solution to the system (1.1) − (1.2) is equivalent to the existence of a fixed point of the operator T.

Lemma 3.1 ([20]). *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold:*
(i) For any $x \in U$, $\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}x(t)$ and $\frac{\log t}{1+(\log t)^{\sigma+2}}D_{1^+}^{p-1}x(t)$ are equi-continuous on any compact interval of $[1,\infty)$.
(ii) For any $\epsilon > 0$, there exists a constant $L = L(\epsilon) > 0$, such that

$$\left|\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}x(t_1) - \frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}x(t_2)\right| < \epsilon,$$

and

$$\left|\frac{\log t_1}{1+(\log t_1)^{\sigma+2}}D_{1^+}^{p-1}x(t_1) - \frac{\log t_2}{1+(\log t_2)^{\sigma+2}}D_{1^+}^{p-1}x(t_2)\right| < \epsilon,$$

for any $t_1, t_2 \geq L$ and $x \in U$.

Lemma 3.2. *Assume that $(A_1)-(A_3)$ hold, then the operator $T : X \times Y \to X \times Y$ is completely continuous.*

Proof. We show that the operator $T : X \times Y \to X \times Y$ is well defined. For any $(x, y) \in X \times Y$, by the definition of operator T, we get $T_1(x, y), D_{1^+}^{p-1}T_1(x, y) \in C[1,\infty)$. From $(A_1)(A_2)$, there exist $\varphi_r(s), \phi_r(s)$, such that

$$\begin{aligned}&|f(s, x(s), D_{1^+}^{q-1}y(s))|\\ =&\left|f(s, \frac{1+(\log s)^{\sigma+2}}{(\log s)^{2-p}}\frac{(\log s)^{2-p}}{1+(\log s)^{\sigma+2}}x(s), \frac{1+(\log s)^{\sigma+2}}{\log s}\frac{\log s}{1+(\log s)^{\sigma+2}}D_{1^+}^{q-1}y(s))\right|\\ \leq&\varphi_r(s),\end{aligned}$$

$$\begin{aligned}&|g(s, x(s), D_{1^+}^{p-1}y(s))|\\ =&\left|g(s, \frac{1+(\log s)^{\sigma+2}}{(\log s)^{2-q}}\frac{(\log s)^{2-q}}{1+(\log s)^{\sigma+2}}x(s), \frac{1+(\log s)^{\sigma+2}}{\log s}\frac{\log s}{1+(\log s)^{\sigma+2}}D_{1^+}^{p-1}y(s))\right|\\ \leq&\phi_r(s),\end{aligned}$$

for $s \in [1,\infty)$. Further,

$$\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}|T_1(x, y)(t)| \leq W_1\int_1^\infty \varphi_r(s)\frac{ds}{s} < \infty,$$

$$\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{p-1}T_1(x, y)(t)| \leq \Pi_1\int_1^\infty \varphi_r(s)\frac{ds}{s} < \infty.$$

Then $T_1(x, y) \in X$. Similarly, we can get $T_2(x, y) \in Y$. Hence, $T : X \times Y \to X \times Y$ is well defined.

Next we prove that T is continuous. Let $(x_n, y_n) \to (x, y)$ as $n \to \infty$ in $X \times Y$, we need to prove that $T_1(x_n, y_n) \to T_1(x, y)$ as $n \to \infty$. It is easy to see that there exists $r > 0$ such that

$$\|(x_n, y_n)\| = \max\{\|x_n\|_X, \|y_n\|_Y\}$$

$$= \max\left\{\sup_{t\in[1,\infty)}\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}|x_n(t)|, \sup_{t\in[1,\infty)}\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{p-1}x_n(t)|\right.$$
$$\left.\sup_{t\in[1,\infty)}\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}|y_n(t)|, \sup_{t\in[1,\infty)}\frac{\log t}{1+(\log t)^{\sigma+2}}|D_{1^+}^{q-1}y_n(t)|\right\}\leq r$$
$$<\infty.$$

Since for $t\in[1,\infty), \|x\|_X\leq r, \|y\|_Y\leq r$,

$$|f(s,x_n(s),D_{1^+}^{q-1}y_n(s))|\leq\varphi_r(s), |g(s,x_n(s),D_{1^+}^{p-1}y_n(s))|\leq\phi_r(s),$$

we have

$$\int_1^\infty\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G(t,s)|f(s,x_n(s),D_{1^+}^{q-1}y_n(s))|\frac{ds}{s}$$
$$\leq W_1\int_1^\infty\varphi_r(s)\frac{ds}{s}<\infty,$$

and

$$\int_1^\infty\frac{\log t}{1+(\log t)^{\sigma+2}}G^*(t,s)|f(s,x_n(s),D_{1^+}^{q-1}y_n(s))|\frac{ds}{s}$$
$$\leq\Pi_1\int_1^\infty\varphi_r(s)\frac{ds}{s}<\infty.$$

So, the continuity of f and the Lebesgue dominated convergence theorem ensure that

$$\lim_{n\to\infty}\int_1^\infty\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G(t,s)f(s,x_n(s),D_{1^+}^{q-1}y_n(s))\frac{ds}{s}$$
$$=\int_1^\infty\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G(t,s)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s},$$

and

$$\lim_{n\to\infty}\int_1^\infty\frac{\log t}{1+(\log t)^{\sigma+2}}G^*(t,s)f(s,x_n(s),D_{1^+}^{q-1}y_n(s))\frac{ds}{s}$$
$$=\int_1^\infty\frac{\log t}{1+(\log t)^{\sigma+2}}G^*(t,s)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}.$$

So

$$\sup_{t\in[1,\infty)}\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}|T_1(x_n,y_n)(t)-T_1(x,y)(t)|\to 0,$$

as $n\to\infty$, which shows the operator T_1 is continuous. In the same way, the operator T_2 is also continuous. Therefore, the operator T is continuous.

Let $I\subset[1,\infty)$ be any compact interval and $U^*=\{(x,y)|(x,y)\in X\times Y, \|(x,y)\| \leq r\}\subseteq X\times Y$ be a bounded set. For any $(x,y)\in U^*, t_1,t_2\in I, t_1<t_2$, we have

$$\left|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}(T_1(x,y))(t_2)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}(T_1(x,y))(t_1)\right|$$
$$=\left|\int_1^\infty\left(\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G(t_2,s)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}G(t_1,s)\right)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}\right|$$

$$
\begin{aligned}
=&\Big|\int_1^\infty \Big(\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}(G_1(t_2,s)+G_2(t_2,s)) \\
&-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}(G_1(t_1,s)+G_2(t_1,s))\Big)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}\Big| \\
\le&\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_2(t_2,s)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}G_2(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
\le&\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_2(t_2,s)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}G_2(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
=&\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}\Big|G_1(t_1,s)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_2(t_2,s)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}G_2(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}.
\end{aligned}
$$

We now consider

$$
\begin{aligned}
&\int_1^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
=&\int_1^{t_1} \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_{t_1}^{t_2} \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\int_{t_2}^\infty \Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_2,s)-\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}G_1(t_1,s)\Big|f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
\le&\frac{1}{\Gamma(p)}\int_1^{t_1}\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}\cdot \\
&\Big[(\log t_2)^{p-1}-(\log t_1)^{p-1}+(\log\frac{t_2}{s})^{p-1}-(\log\frac{t_1}{s})^{p-1}\Big]f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\frac{1}{\Gamma(p)}\int_{t_1}^{t_2}\Big|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}\cdot\Big[(\log t_2)^{p-1}-(\log t_1)^{p-1}+(\log\frac{t_2}{s})^{p-1}\Big]f(s,x(s), \\
&D_{1^+}^{q-1}y(s))\frac{ds}{s} \\
&+\frac{1}{\Gamma(p)}\int_{t_2}^\infty\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}\cdot\Big[(\log t_2)^{p-1}-(\log t_1)^{p-1}\Big]f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}. \\
\to&\ 0,\ t_1\to t_2.
\end{aligned}
$$

We conclude

$$\left|\frac{(\log t_2)^{2-p}}{1+(\log t_2)^{\sigma+2}}(T_1(x,y))(t_2)-\frac{(\log t_1)^{2-p}}{1+(\log t_1)^{\sigma+2}}(T_1(x,y))(t_1)\right|\to 0,$$

as $t_1 \to t_2$. Therefore, $\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}(T_1(x,y))(t)$ is equi-continuous on I. Note that

$$D_{1^+}^{p-1}(T_1(x,y))(t)=\int_1^\infty G^*(t,s)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}.$$

Since $D_{1^+}^{p-1}(T_1(x,y))(t)$ does not depend on t, we conclude

$$\left|\frac{\log t_2}{1+(\log t_2)^{\sigma+2}}D_{1^+}^{p-1}(T_1(x,y))(t_2)-\frac{\log t_1}{1+(\log t_1)^{\sigma+2}}D_{1^+}^{p-1}(T_1(x,y))(t_1)\right|\to 0,$$

as $t_1 \to t_2$. Therefore, $\frac{\log t}{1+(\log t)^{\sigma+2}}D_{1^+}^{p-1}(T_1(x,y))(t)$ is equi-continuous on I.

By using the same way, we can show that $\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}}(T_2(x,y))(t)$, $\frac{\log t}{1+(\log t)^{\sigma+2}}D_{1^+}^{q-1}(T_2(x,y))(t)$ are equi-continuous. Thus T_1, T_2 are equi-continuous on I. So the operator T is equi-continuous for $(x,y)\in U^*$ on any interval $I\times I$ of $[1,\infty)$.

Finally, we prove that T_1 is equi-continuous at ∞. For any $(x,y)\in U^*$,

$$\begin{aligned}&\lim_{t\to\infty}\left|\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}(T_1(x,y))(t)\right|\\&\quad=\lim_{t\to\infty}\int_1^\infty\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}}G(t,s)|f(s,x(s),D_{1^+}^{q-1}y(s))|\frac{ds}{s}\\&\quad\le W_1\int_1^\infty\varphi_r(s)\frac{ds}{s}<\infty.\end{aligned}$$

In addition, for any $(x,y)\in U^*$, we have

$$\begin{aligned}&\lim_{t\to\infty}\left|\frac{\log t}{1+(\log t)^{\sigma+2}}D_{1^+}^{p-1}(T_1(x,y))(t)\right|\\&\quad=\lim_{t\to\infty}\int_1^\infty\frac{\log t}{1+(\log t)^{\sigma+2}}G^*(t,s)|f(s,x(s),D_{1^+}^{q-1}y(s))|\frac{ds}{s}\\&\quad\le \Pi_1\int_1^\infty\varphi_r(s)\frac{ds}{s}<\infty.\end{aligned}$$

So, T_1 is equi-continuous at ∞. Similarly, T_2 is equi-continuous at ∞. Therefore, by Lemma 3.1, we conclude that $T: X\times Y\to X\times Y$ is a completely continuous operator. □

Theorem 3.1. *Assume that* $(A_1)-(A_3)$ *are satisfied. In addition:*
(A_4) *there exist nonnegative functions* $a(t), b(t), c(t), d(t), e(t), l(t)$ *with*

$$a^*=\int_1^\infty a(t)\frac{dt}{t}<\infty, b^*=\int_1^\infty b(t)\frac{dt}{t}<\infty, c^*=\int_1^\infty c(t)\frac{dt}{t}<\infty,$$

$$d^*=\int_1^\infty d(t)\frac{dt}{t}<\infty, e^*=\int_1^\infty e(t)\frac{dt}{t}<\infty, l^*=\int_1^\infty l(t)\frac{dt}{t}<\infty,$$

such that

$$|f(t,x,y)| \le a(t) + \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} b(t)|x| + \frac{\log t}{1+(\log t)^{\sigma+2}} c(t)|y|,$$

$$|g(t,x,y)| \le d(t) + \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} e(t)|x| + \frac{\log t}{1+(\log t)^{\sigma+2}} l(t)|y|,$$

for $t \in [1,\infty), x, y \in \boldsymbol{R}$. *If*

$$\max\{(b^* + c^*)W_1, (e^* + l^*)W_2\} < 1,$$

then the system (1.1) − (1.2) *has at least one nontrivial solution* $(x(t), y(t)), t \in [1,\infty)$.

Proof. Let the set $U^* = \{(x,y) \in X \times Y, \|(x,y)\| \le r\}$, where the positive number r satisfies

$$r \ge \max\left\{ \frac{a^* W_1}{1-(b^*+c^*)W_1}, \frac{d^* W_2}{1-(e^*+l^*)W_2} \right\}. \tag{3.1}$$

We show firstly that $TU^* \subset U^*$. For $(x,y) \in U^*, t \in [1,\infty)$, by using Lemma 2.4, we obtain

$$\begin{aligned}
&\left| \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} (T_1(x,y))(t) \right| \\
&= \left| \int_1^\infty \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G(t,s) f(s, x(s), D_{1^+}^{q-1} y(s)) \frac{ds}{s} \right| \\
&\le W_1 \int_1^\infty |f(s, x(s), D_{1^+}^{q-1} y(s))| \frac{ds}{s} \\
&\le W_1 \int_1^\infty \left[a(s) + \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} b(s)|x(s)| + \frac{\log t}{1+(\log t)^{\sigma+2}} c(s) D_{1^+}^{q-1} |y(s)| \right] \frac{ds}{s} \\
&\le W_1 \int_1^\infty a(s) \frac{ds}{s} + W_1 r \int_1^\infty b(s) \frac{ds}{s} + W_1 r \int_1^\infty c(s) \frac{ds}{s} \\
&= W_1 (a^* + rb^* + rc^*),
\end{aligned}$$

and

$$\begin{aligned}
&\left| \frac{\log t}{1+(\log t)^{\sigma+2}} D_{1^+}^{p-1} (T_1(x,y))(t) \right| \\
&= \left| \int_1^\infty \frac{\log t}{1+(\log t)^{\sigma+2}} G^*(t,s) f(s, x(s), D_{1^+}^{q-1} y(s)) \frac{ds}{s} \right| \\
&\le \Pi_1 \int_1^\infty |f(s, x(s), D_{1^+}^{q-1} y(s))| \frac{ds}{s} \\
&\le \Pi_1 \int_1^\infty \left[a(s) + \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} b(s)|x(s)| + \frac{\log t}{1+(\log t)^{\sigma+2}} c(s) D_{1^+}^{q-1} |y(s)| \right] \frac{ds}{s} \\
&\le \Pi_1 \int_1^\infty a(s) \frac{ds}{s} + \Pi_1 r \int_1^\infty b(s) \frac{ds}{s} + \Pi_1 r \int_1^\infty c(s) \frac{ds}{s} \\
&= \Pi_1 (a^* + rb^* + rc^*).
\end{aligned}$$

In addition,

$$\begin{aligned}W_1-\Pi_1&=\frac{1}{\Gamma(p)}\left[1+\frac{1+\Gamma(p)}{(1-\Delta_1)\Gamma(p)}\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}\right]-1-\frac{1}{1-\Delta_1}\sum_{i=1}^{m}a_iG_1(\eta_i,s)\\&\geq\frac{1}{\Gamma(p)}-1+\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}\left[\frac{1+\Gamma(p)}{(1-\Delta_1)\Gamma^2(p)}-\frac{1}{(1-\Delta_1)\Gamma(p)}\right]\\&=\frac{1}{\Gamma(p)}-1+\sum_{i=1}^{m}a_i(\log\eta_i)^{p-1}\left[\frac{1}{(1-\Delta_1)\Gamma^2(p)}\right]\geq 0.\end{aligned}$$

So $\|T_1(x,y)\|_X\leq W_1(a^*+rb^*+rc^*)$. In a similar manner, we have $\|T_2(x,y)\|_Y\leq W_2(d^*+re^*+rl^*)$. Therefore, by using condition (3.1), we obtain $\|T(x,y)\|_{X\times Y}\leq r$.

By Lemma 3.2, we have $T:U^*\to U^*$ is completely continuous. U^* is a nonempty, closed, bounded and convex subset of $X\times Y$. According to the Schauder fixed point theorem, T has at least one fixed point, so the system (1.1)-(1.2) has at least one solution.

Assume there is a trivial solution $(0,0)$ to $(1.1)-(1.2)$, that is, the operator T has a fixed point $(0,0)$. Since f,g are continuous, $f(t,0,0),g(t,0,0)\not\equiv 0$ on any subinterval of $t\in[1,\infty),G(t,s),H(t,s)$ are nonnegative and continuous for $(t,s)\in[1,\infty)\times[1,\infty)$, we get

$$T_1(x,y)(t)=\int_1^\infty G(t,s)f(s,x(s),D_{1^+}^{q-1}y(s))\frac{ds}{s}\neq 0,$$

$$T_2(x,y)(t)=\int_1^\infty H(t,s)g(s,x(s),D_{1^+}^{p-1}y(s))\frac{ds}{s}\neq 0.$$

Thus, $(1.1)-(1.2)$ has at least one nontrivial solution. □

Theorem 3.2. *Assume that* $(A_1)-(A_3)$ *hold, in addition:*
(A_5) *there exist nonnegative functions* $a'(t),b'(t),c'(t),d'(t)$ *with*

$$a'^*=\int_1^\infty a'(t)\frac{dt}{t}<\infty, b'^*=\int_1^\infty b'(t)\frac{dt}{t}<\infty,$$

$$c'^*=\int_1^\infty c'(t)\frac{dt}{t}<\infty, d'^*=\int_1^\infty d'(t)\frac{dt}{t}<\infty,$$

such that

$$\begin{aligned}&|f(t,x_1,D_{1^+}^{q-1}y_1)-f(t,x_2,D_{1^+}^{q-1}y_2)|\\\leq&\frac{(\log t)^{2-p}a'(t)}{1+(\log t)^{\sigma+2}}|x_1-x_2|+\frac{(\log t)b'(t)}{1+(\log t)^{\sigma+2}}|D_{1^+}^{q-1}y_1-D_{1^+}^{q-1}y_2|,\end{aligned}$$

$$\begin{aligned}&|g(t,x_1,D_{1^+}^{p-1}y_1)-g(t,x_2,D_{1^+}^{p-1}y_2)|\\\leq&\frac{(\log t)^{2-q}c'(t)}{1+(\log t)^{\sigma+2}}|x_1-x_2|+\frac{(\log t)d'(t)}{1+(\log t)^{\sigma+2}}|D_{1^+}^{p-1}y_1-D_{1^+}^{p-1}y_2|,\end{aligned}$$

for $t\in[1,\infty),x_1,x_2,y_1,y_2\in\boldsymbol{R}$. *If*

$$\Lambda_1:=\max\{W_1a'^*,W_1b'^*\},\Lambda_2:=\max\{W_2c'^*,W_2d'^*\},$$

$$\Lambda := \max\{\Lambda_1, \Lambda_2\} = \max\{W_1 a'^*, W_1 b'^*, W_2 c'^*, W_2 d'^*\} < 1,$$

then the system (1.1) − (1.2) *has a unique solution* $(x^*, y^*) \in X \times Y$. *In addition, for* $(x_0, y_0) \in X \times Y$, *the sequence* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *defined by*

$$\begin{aligned}&(x_n, y_n)\\ =&\Big(\int_1^\infty G(t,s) f(s, x_n(s), D_{1^+}^{q-1} y_n(s)) \frac{ds}{s}, \int_1^\infty H(t,s) g(s, x_n(s), D_{1^+}^{p-1} y_n(s)) \frac{ds}{s}\Big)\end{aligned}$$

converges to (x^*, y^*), *as* $n \to \infty$, *and we have the following error estimate:*

$$\|(x_n, y_n) - (x^*, y^*)\| \le \frac{\Lambda^n}{1-\Lambda} \|(x_1, y_1) - (x^*, y^*)\|.$$

Proof. By using Lemma 2.4 and (A_5), for $(x_1, y_1), (x_2, y_2) \in X \times Y$, we obtain

$$\begin{aligned}
&\Big|\frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} T_1(x_1,y_1)(t) - \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} T_1(x_2,y_2)(t)\Big|\\
=&\Big|\int_1^\infty \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G(t,s) f(t, x_1(s), D_{1^+}^{q-1} y_1(s)) \frac{ds}{s}\\
&-\int_1^\infty \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G(t,s) f(t, x_2(s), D_{1^+}^{q-1} y_2(s)) \frac{ds}{s}\Big|\\
\le&\int_1^\infty \frac{(\log t)^{2-p}}{1+(\log t)^{\sigma+2}} G(t,s) |f(t, x_1(s), D_{1^+}^{q-1} y_1(s)) - f(t, x_2(s), D_{1^+}^{q-1} y_2(s))| \frac{ds}{s}\\
\le&W_1 \int_1^\infty \Big(\frac{(\log t)^{2-p} a'(t)}{1+(\log t)^{\sigma+2}} |x_1(s) - x_2(s)| + \frac{(\log t)^{2-p} b'(t)}{1+(\log t)^{\sigma+2}} |D_{1^+}^{q-1} y_1(s) - D_{1^+}^{q-1} y_2(s)|\Big) \frac{ds}{s}\\
=&W_1 (a'^* \|x_1 - x_2\|_X + b'^* \|y_1 - y_2\|_Y)\\
\le&\max\{W_1 a'^*, Y_1 b'^*\}(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y)\\
=&\Lambda_1 \|(x_1, y_1) - (x_2, y_2)\|, t \in [1, \infty).
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
&\Big|\frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} T_2(x_1,y_1)(t) - \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} T_2(x_2,y_2)(t)\Big|\\
=&\Big|\int_1^\infty \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} H(t,s) g(t, x_1(s), D_{1^+}^{p-1} y_1(s)) \frac{ds}{s}\\
&-\int_1^\infty \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} H(t,s) g(t, x_2(s), D_{1^+}^{p-1} y_2(s)) \frac{ds}{s}\Big|\\
\le&\int_1^\infty \frac{(\log t)^{2-q}}{1+(\log t)^{\sigma+2}} H(t,s) |g(t, x_1(s), D_{1^+}^{p-1} y_1(s)) - g(t, x_2(s), D_{1^+}^{p-1} y_2(s))| \frac{ds}{s}\\
\le&W_2 \int_1^\infty \Big(\frac{(\log t)^{2-q} c'(t)}{1+(\log t)^{\sigma+2}} |x_1(s) - x_2(s)| + \frac{(\log t)^{2-q} d'(t)}{1+(\log t)^{\sigma+2}} |D_{1^+}^{p-1} y_1(s) - D_{1^+}^{p-1} y_2(s)|\Big) \frac{ds}{s}\\
=&W_2 (c'^* \|x_1 - x_2\|_X + d'^* \|y_1 - y_2\|_Y)\\
\le&\max\{W_2 c'^*, W_2 d'^*\}(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y)\\
=&\Lambda_2 \|(x_1, y_1) - (x_2, y_2)\|,\ t \in [1, \infty).
\end{aligned}$$

From the above inequalities, we deduce

$$\begin{aligned}
&\|T(x_1, y_1) - T(x_2, y_2)\|\\
=&\max\{\|T_1(x_1, y_1) - T_1(x_2, y_2)\|, \|T_2(x_1, y_1) - T_2(x_2, y_2)\|\}
\end{aligned}$$

$$\le \max\{\Lambda_1,\Lambda_2\}\|(x_1,y_1)-(x_2,y_2)\| \\ =\Lambda\|(x_1,y_1)-(x_2,y_2)\|.$$

Thus, T is a contraction operator. By using the Banach contraction mapping principle, we conclude that T has a unique fixed point $(x^*,y^*)\in X\times Y$, which is the unique solution of the system (1.1) − (1.2). In addition, for $(x_0,y_0)\in X\times Y$, the sequence $\{(x_n,y_n)\}_{n=1}^{\infty}$ defined by $(x_n,y_n)=(\int_1^\infty G(t,s)f(s,x_n(s),D_{1^+}^{q-1}y_n(s))\frac{ds}{s},\int_1^\infty H(t,s)g(s,x_n(s),D_{1^+}^{p-1}y_n(s))\frac{ds}{s})$ converges to (x^*,y^*), as $n\to\infty$. From the proof of the Banach fixed point theorem, we obtain the error estimate. □

4. Examples

Consider the following equations:

$$\begin{cases} D_{1^+}^{\frac{3}{2}}x(t)+f(t,x(t),D_{1^+}^{\frac{1}{2}}y(t))=0, t\in(1,\infty), \\ D_{1^+}^{\frac{3}{2}}y(t)+g(t,x(t),D_{1^+}^{\frac{1}{2}}y(t))=0, t\in(1,\infty), \end{cases} \tag{4.1}$$

supplemented with

$$\begin{cases} \lim_{t\to1}(\log t)^{\frac{1}{2}}x(t)=\lim_{t\to\infty}D_{1^+}^{\frac{1}{2}}x(t)=\sum_{i=1}^{2}a_ix(\eta_i), \\ \lim_{t\to1}(\log t)^{\frac{1}{2}}y(t)=\lim_{t\to\infty}D_{1^+}^{\frac{1}{2}}y(t)=\sum_{j=1}^{2}b_jy(\zeta_j), \end{cases} \tag{4.2}$$

where $p=q=\frac{3}{2}, m=n=2, a_1=a_2=\frac{1}{10}, b_1=b_2=\frac{1}{12}, \eta_1=e, \eta_2=e^2, \zeta_1=e, \zeta_2=e^3$.

$$\triangle_1=1-\sum_{i=1}^{2}a_i(\frac{(\log\eta_i)^{p-1}}{\Gamma(p)}+(\log\eta_i)^{p-2})\approx0.44>0,$$

$$\triangle_2=1-\sum_{j=1}^{2}b_j(\frac{(\log\zeta_j)^{q-1}}{\Gamma(q)}+(\log\zeta_j)^{q-2})\approx0.39>0.$$

Example 4.1. Consider the following functions:

$$f(t,x,y)=\frac{(\log t)^{\frac{1}{2}}te^{-2t}x}{15(2+(\log t)^2)}+\frac{(\log t)^2t^{-2}y}{6(1+(\log t)^2)}+\frac{1}{t},$$

$$g(t,x,y)=\frac{(\log t)^{\frac{1}{2}}te^{-3t}x}{10(1+(\log t)^2)}+\frac{(\log t)^2t^{-1}y}{12(2+(\log t)^2)}+\frac{1}{t^2}.$$

Let $\sigma=0$. Choose

$$a(t)=\frac{1}{t}, b(t)=\frac{te^{-2t}}{15}, c(t)=\frac{t^{-2}\log t}{6}, d(t)=\frac{1}{t^2}, e(t)=\frac{te^{-3t}}{10}, l(t)=\frac{t^{-1}\log t}{12}.$$

By simple calculation, we have

$$a^*=\int_1^\infty\frac{1}{t}\frac{dt}{t}=1, b^*=\int_1^\infty\frac{te^{-2t}}{15}\frac{dt}{t}=\frac{1}{30e^2}, c^*=\int_1^\infty\frac{t^{-2}\log t}{6}\frac{dt}{t}=\frac{1}{24},$$

$$d^*=\int_1^\infty\frac{1}{t^2}\frac{dt}{t}=\frac{1}{2}, e^*=\int_1^\infty\frac{te^{-3t}}{10}\frac{dt}{t}=\frac{1}{30e^3}, l^*=\int_1^\infty\frac{t^{-1}\log t}{12}\frac{dt}{t}=\frac{1}{12}.$$

For the functions f and g, the assumptions $(A_1)(i),(ii),(A_2)(i),(ii)$ and (A_3) are readily verified. The functions f and g satisfy the inequalities

$$|f(t,x,y)| \le \frac{(\log t)^{\frac{1}{2}}b(t)|x|}{1+(\log t)^2} + \frac{(\log t)c(t)|y|}{1+(\log t)^2} + a(t),$$

$$|g(t,x,y)| \le \frac{(\log t)^{\frac{1}{2}}e(t)|x|}{1+(\log t)^2} + \frac{(\log t)l(t)|y|}{1+(\log t)^2} + d(t).$$

In addition, for $t \in [1,\infty), |x|,|y| \le r$, we find

$$f(t, \frac{1+(\log t)^2}{(\log t)^{\frac{1}{2}}}x, \frac{1+(\log t)^2}{\log t}y) \le b(t)r + c(t)r + a(t) =: \varphi_r(t),$$

$$g(t, \frac{1+(\log t)^2}{(\log t)^{\frac{1}{2}}}x, \frac{1+(\log t)^2}{\log t}y) \le e(t)r + l(t)r + d(t) =: \phi_r(t),$$

so

$$\int_1^\infty \varphi_r(t)\frac{dt}{t} = b^*r + c^*r + a^* < \infty, \int_1^\infty \phi_r(t)\frac{dt}{t} = e^*r + l^*r + d^* < \infty,$$

that is to say, assumptions $(A_1)(iii)$ and $(A_2)(iii)$ are satisfied. Because

$$\max\{(b^*+c^*)W_1, (e^*+l^*)W_2\} \approx 0.31 < 1,$$

by Theorem 3.1, the system $(4.1)-(4.2)$ has at least one nontrivial solution.

Example 4.2. Consider the following functions:

$$f(t,x,y) = \frac{(\log t)^{\frac{1}{2}}te^{-2t}}{10(1+(\log t)^2)}\sqrt{x+1} + \frac{(\log t)^2t^{-2}}{20(1+(\log t)^2)}\sin(y+1) + \frac{1}{t},$$

$$g(t,x,y) = \frac{(\log t)^{\frac{1}{2}}te^{-3t}}{20(1+(\log t)^2)}\arctan x + \frac{(\log t)^2t^{-1}}{10(1+(\log t)^2)}\cos(y+1) + \frac{1}{t^2}.$$

For the functions f and g, the assumptions $(A_1)(i),(ii),(A_2)(i),(ii)$ and (A_3) are readily verified. Let $\sigma = 0$. Choose

$$a'(t) = \frac{te^{-2t}}{10}, b'(t) = \frac{t^{-2}\log t}{20}, c'(t) = \frac{te^{-3t}}{20}, d'(t) = \frac{t^{-1}\log t}{10}.$$

The functions f and g satisfy the inequalities

$$\begin{aligned}&|f(t,x_1,D_{1^+}^{q-1}y_1) - f(t,x_2,D_{1^+}^{q-1}y_2)|\\ \le&\frac{(\log t)^{2-p}a'(t)}{1+(\log t)^{\sigma+2}}|x_1-x_2| + \frac{(\log t)b'(t)}{1+(\log t)^{\sigma+2}}|D_{1^+}^{q-1}y_1 - D_{1^+}^{q-1}y_2|,\end{aligned}$$

$$\begin{aligned}&|g(t,x_1,D_{1^+}^{p-1}y_1) - g(t,x_2,D_{1^+}^{p-1}y_2)|\\ \le&\frac{(\log t)^{2-q}c'(t)}{1+(\log t)^{\sigma+2}}|x_1-x_2| + \frac{(\log t)d'(t)}{1+(\log t)^{\sigma+2}}|D_{1^+}^{p-1}y_1 - D_{1^+}^{p-1}y_2|.\end{aligned}$$

By a simple calculation, we have

$$a'^* = \int_1^\infty \frac{te^{-2t}}{10}\frac{dt}{t} = \frac{1}{20e}, b'^* = \int_1^\infty \frac{t^{-2}\log t}{20}\frac{dt}{t} = \frac{1}{80},$$

$$c'^* = \int_1^\infty \frac{te^{-3t}}{20}\frac{dt}{t} = \frac{1}{60e}, d'^* = \int_1^\infty \frac{t^{-1}\log t}{10}\frac{dt}{t} = \frac{1}{10}.$$

In addition, for $t \in [1, \infty), |x|, |y| \leq r$, we find

$$f(t, \frac{1 + (\log t)^2}{(\log t)^{\frac{1}{2}}}x, \frac{1 + (\log t)^2}{\log t}y) \leq a'(t)(1 + r) + b'(t)(1 + r) + a(t) =: \varphi_r(t),$$

$$g(t, \frac{1 + (\log t)^2}{(\log t)^{\frac{1}{2}}}x, \frac{1 + (\log t)^2}{\log t}y) \leq c'(t)r + d'(t)(1 + r) + d(t) =: \phi_r(t),$$

so

$$\int_1^\infty \varphi_r(t)\frac{dt}{t} = a'^*(1 + r) + b'^*(1 + r) + a^* < \infty,$$

$$\int_1^\infty \phi_r(t)\frac{dt}{t} = c'^*r + d'^*(1 + r) + d^* < \infty,$$

that is to say, assumptions $(A_1)(iii)$ and $(A_2)(iii)$ are satisfied. We also obtain $W_1 \approx 6.86, W_2 \approx 2.53, \Lambda \approx 0.25 < 1$. Hence, by Theorem 3.2, we conclude that $(4.1) - (4.2)$ has a unique solution (x^*, y^*).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Availability of data and materials

Not applicable.

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