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On Constacyclic Codes of Length $4p^s$ over $F_{p^m}[u,v]/\langle u^2, v^2, uv-vu\rangle$

Somaiyah A. A. Abdulsattar[1,†] and Arunkumar Patil[2]

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Abstract In this paper, we determine the algebraic structure of all λ- constacyclic codes of length $4p^s$ over the ring $R_{u^2,v^2,p^m} = F_{p^m}[u,v]/\langle u^2, v^2, uv-vu\rangle$ and $u^2=0, v^2=0$ where $\lambda=(\alpha+\beta u+\gamma v+\delta uv)$ with β,γ, $\delta\in F_{p^m}$, $\alpha\in F^*_{p^m}$ and β, γ are not both zero. If λ is a square, each λ-constacyclic codes of length $4p^s$ is expressed as a direct sum of an -α-constacyclic code and α-constacyclic code of length $2p^s$. In the primary case where the unit λ is not square, it is shown that any non-zero polynomial of degree ≤ 4 over $F_{p^m}$is invertible in the ring $R_{\alpha,\beta,\gamma,\delta} = R_{u^2,v^2,p^m}[x]/\langle x^{4p^s}-\lambda\rangle$ when $\lambda=(\alpha+\beta u+\gamma v+\delta uv)$ for non-zero elements α, β,$\gamma\in F^*_{p^m}$, $\delta\in F_{p^m}$, it follows that the ring $R_{\alpha,\beta,\gamma,\delta}$ is a chain ring with maximal ideal $(x^4-\alpha_0)$ and $(\alpha+\beta u+\gamma v+\delta uv)$- constacyclic codes are $\langle(x^4-\alpha_0)^i\rangle$, for $0\leq i\leq 2p^s$. In the second case where λ is not square and $\lambda=\gamma$ for $\gamma\in F_{p^m}$, it is obtain that $R_{u^2,v^2,p^m}[x]/\langle x^{4p^s}-\gamma\rangle$ is a local ring with maximal ideal $\langle x^4-\gamma_0, u\rangle$ but not a chain ring, such λ-constacyclic codes are classified into four distinct types. We provide the number of codewords for each type, and give the details of their dual codes.

Keywords Constacyclic codes, codes over rings, dual codes

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1. Introduction

In the realm of error-correcting code theory, constacyclic codes are significant due to their efficient encoding capabilities using simple shift registers. These codes possess robust algebraic structures that facilitate effective error correction and detection, making them essential for both practical applications and theoretical explorations. However, the classification of these codes, particularly for lengths such as $4p^s$, presents a complex challenge. Historically, only specific classes of constacyclic codes of certain lengths over particular finite rings have been successfully classified. Let R denote a finite commutative ring with unity, and let λ be a unit of R. A λ-constacyclic code of length n over R can be represented as an ideal of the quotient ring $R[x]/\langle x^n-\lambda\rangle$. When the code length n is coprime to the characteristic of the residue field of R, these codes are categorized as simple root codes, otherwise, they are referred to as repeated root codes.

[†] the corresponding author.
Email address: somaia_alshamiry2013@yahoo.com (S. A. A. Abdulsattar), arpatil@sggs.ac.in (A. Patil)
[1] School of Mathematics Sciences, SRTM University, Vishnupuri, Nanded, 431606, Maharashtra, India.
[2] Department of Mathematics, SGGS Institute of Engineering and Technology, Vishnupuri, Nanded, 431606, Maharashtra, India.

Recent years have seen a surge of interest in codes over finite rings, leading to the discovery of several effective codes as Gray images of codes over rings. Notably, special classes of simple and repeated root constacyclic codes over various finite rings have been extensively studied. For instance, see [1–15]. Yildiz et al. [16] studied the ring $F_2[u,v]/ < u^2, v^2, uv - vu >$ which is not a chain ring and examined cyclic codes of odd length over it, and identified effective binary codes as Gray images of cyclic codes. A generalization of these codes was explored in [17], where the authors investigated the cyclic codes over $F_2[u_1, u_2, ..., u_k]/ < u_i^2, u_j^2, u_i u_j - u_j u_i >$. Subsequently, Sobhani and Molakarimi [18] extended these studies to cyclic codes over the ring $F_{2^m}[u,v]/ < u^2, v^2, uv - vu >$. In addition, in 2015, Kewat et al. [19] investigated cyclic codes over the ring $Z_p[u,v]/ < u^2, v^2, uv - vu >$.

Rings of these forms have been used as alphabets of certain cyclic codes by numerous authors [20, 21]. Dinh et al. [22] determined the algebraic structures of all constacyclic codes of length p^s over $R_{u,v}$, except for the $(\alpha + \delta uv)$ - constacyclic codes where $\delta \neq 0$ which are studied in [23]. In the case $p = 2$, Dinh et al. classified all self-dual λ-constacyclic codes of length 2^s over $R_{u,v}$. Y. Ahendouz and I. Akharraa [26] studied some cases of constacyclic codes of length $2p^s$ over $R_{u,v}$.

Motivated by the above-mentioned works and to our knowledge, no notable work yet that conducts that research has been published on extending the classification of constacyclic codes to the length $4p^s$ over the ring $R_{u,v} = F_{p^m}[u,v]/ < u^2, v^2, uv - vu >$, where p is an odd prime number. This ring serves as a crucial example of a finite local ring that is not a chain ring. Its unit has the form $\lambda = (\alpha + \beta u + \gamma v + \delta uv)$ with α, β,γ, $\delta \in F_{p^m}$, $\alpha \in F^*_{p^m}$. Our investigation will build upon previous works, particularly focusing on the algebraic structures of λ- constacyclic codes of this new length, thereby contributing to the ongoing discourse in the field of error-correcting codes.

The rest of this paper is organized as follows: Section 2 provides fundamental background on constacyclic codes. Section 3 investigates λ-constacyclic codes of length $4p^s$ over $R_{u,v}$, focusing on the number of elements in each code and their duals when λ is a square. Section 4 extends this analysis to λ-constacyclic codes in scenarios where λ is not a square. Lastly, section 5 examines the duals of the codes discussed in Section 4.

2. Preliminaries

A ring R is a principal ideal ring if its ideals are principal. An ideal I of R is called principal if it is generated by one element. Let R be a finite commutative ring. If the ring R has a unique maximal ideal then R is called local ring with unity. Also if this unique maximal ideal is principal then R is a chain ring.

Definition 2.1. [25] The socle of a ring R is denoted by $Soc(R)$, defined as the sum of all minimal left ideals of R. A minimal left ideal is an ideal that is not the zero ideal and does not contain any other non-zero left ideals.

Definition 2.2. [29] The Jacobson radical of R is defined to be $J(R)$, which is the intersection of all maximal left ideals of R. If R satisfies the isomorphism

$$\frac{R}{J(R)} \cong Soc(R),$$

then R is called a Frobenius ring.

A linear code C of length n is an $R-$ sub module of R^n. Any element of C is called a codeword.

Definition 2.3. Let λ be a unit of R. A linear code C of length n over R is defined as a λ-constacyclic code if the following holds:

$$(c_0,c_1,...,c_{n-1})\in C\Rightarrow(\lambda c_{n-1},c_0,...,c_{n-2})\in C,$$

where $c=(c_0,c_1,...,c_{n-1})\in C$ can be represented as a polynomial

$$c(x)=c_0+c_1x+...+c_{n-1}x^{n-1}.$$

This leads us to the following proposition:

Proposition 2.1. *[30] A subset C of R^n is a λ-constacyclic code if and only if C is an ideal of $R[x]/\langle x^n-\lambda\rangle$ in view this proposition. We identify for a λ -constacyclic code C as an ideal of $R[x]/\langle x^n-\lambda\rangle$.*

For n-tuples $a=(a_0,a_1,...,a_{n-1}),b=(b_0,b_1,...,b_{n-1})\in R^n$, their inner product is defined as

$$a.b=a_0b_0+a_1b_1+...+a_{n-1}b_{n-1}.$$

Definition 2.4. Any two n-tuples a and b are said to be orthogonal if their inner product $a.b=0$. The dual code $C^\perp$ of a linear code C over a ring R is defined as

$$C^\perp=\{a\in R^n: a.b=0,\ \text{for all}\ \ b\in C\}.$$

A code C is called self orthogonal if $C\subseteq C^\perp$ and a code C is called self dual if $C=C^\perp$.

Proposition 2.2. *[22] A linear code C of length n over a finite Frobenius ring R satisfies $|C^\perp||C|=|R|^n$. The annihilator of an ideal I in the ring R is defined as*

$$A(I)=\{a\in R|a.b=0,\ \ \textit{for all}\ \ b\in I\}.$$

It is denoted as $A(I)$.

For any polynomial $f(x)=a_0+a_1x+...+a_lx^l\in R[x]$ where $a_l\neq 0$, we define the reciprocal polynomial $f^*(x)$ as $f^*(x)=x^la(x^{-1})=a_l+a_{l-1}+...+a_0x^l$.

Proposition 2.3. *[27] The dual of a λ- constacyclic code over R is a λ^{-1}-constacyclic code over R.*

Proposition 2.4. *[5] Let C be a λ-constacyclic code of length n over R. Then*

$$C^\perp=A(C)^*=\{f^*(x)\mid f(x)\in A(C)\}.$$

Let p be an odd prime number and m a positive integer. Consider the following rings:

$$R_{u,v}=\frac{F_{p^m}[u,v]}{\langle u^2,v^2,uv-vu\rangle},$$

and

$$K_v=\frac{F_{p^m}[v]}{\langle v^2\rangle}.$$

It is easy to verify that

$$J(R(_{u,v})) = \langle u, v\rangle \text{ and } Soc(R(_{u,v})) = \langle uv\rangle.$$

As a result we have the isomorphism

$$\tfrac{R_{u,v}}{J(R_{u,v})} \cong Soc(R_{u,v}),$$

which implies that $R_{u,v}$ is a Frobenius ring while K_v is a chain ring with maximal ideal $\langle v\rangle$.

An element of $R_{u,v}$ of the form $\lambda = (\alpha + \beta u + \gamma v + \delta uv)$, where α, β,γ,$\delta \in F^*_{p^m}$ is a unit in F_{p^m}. Let S be a positive integer. By applying the division algorithm, we can express S as

$$S = q_s m + r_s,$$

where $0 \leq r_s \leq m-1$ and q_s, r_s are non-negative integers.

Note that $\alpha^{p^m} = \alpha$

$$\alpha_0 = \alpha^{p^{m-r_s}} = \alpha^{(q_s+1)m-s},$$
$$\alpha_0^{p^s} = \alpha^{p^{(q_s+1)m}} = \alpha.$$

Now we can define rings

$$R_\lambda = \frac{R_{u,v}[x]}{\langle x^{4p^s} - \lambda\rangle},$$

and

$$S_{\alpha+\gamma v} = \frac{K_v[x]}{\langle x^{4p^s} - (\alpha + \gamma v)\rangle}.$$

Consider the homorphism:

$$\Phi: \ R_{u,v} \longrightarrow K_v,$$
$$a_1 + a_2 u + a_3 v + a_4\, uv \longmapsto a_1 + a_3 v,$$

where $a_i \in F_{p^m}$. This homomorphism corresponds to reduction modulo u and naturally extends to a homomorphism between polynomials rings.

$$\phi : R_{u,v}[x] \longrightarrow K_v[x]$$

Since

$$\phi(x^{4p^s} - \lambda) = x^{4p^s} - (\alpha + \gamma v),$$

then

$$\phi : R_\lambda \longrightarrow S_{\alpha+\gamma v}.$$

Proposition 2.5. *[26] The following conditions are equivalent :*

1. *λ is a square in $R_{u,v}$.*
2. *$\alpha + \gamma v$ is a square in K_v.*
3. *α is a square in F_{p^m}.*

3. The unit λ is a square in $R_{u,v}$

In this section we assume λ is a square in the ring $R_{u,v}, (\lambda = \alpha^2)$. We will define two polynomials in $R_{u,v}[x]$.

$$f(x) = (2\alpha)^{-1}(x^{2p^s} + \alpha) \text{ and } g(x) = -(2\alpha)^{-1}(x^{2p^s} - \alpha).$$

In R_λ the following equations hold:

$$\begin{cases} f(x) + g(x) = 1, \\ f(x).g(x) = 0, \\ f(x)^2 = f(x), \\ g(x)^2 = g(x). \end{cases} \tag{3.1}$$

Lemma 3.1. *1.* $R_\lambda = f(x)R_\lambda \oplus g(x)R_\lambda$.

2. The ring homomorphisms

$$\begin{aligned} \phi_f : \frac{R_{u,v}[x]}{\langle x^{2p^s} - \alpha\rangle} &\longrightarrow f(x)R_\lambda \\ z(x) &\longrightarrow f(x)z(x), \end{aligned}$$

and

$$\begin{aligned} \phi_g : \frac{R_{u,v}[x]}{\langle x^{2p^s} + \alpha\rangle} &\longrightarrow g(x)R_\lambda \\ z(x) &\longrightarrow g(x)z(x). \end{aligned}$$

are ring isomorphisms.

Proof.

1. For any $z(x) \in R_\lambda$

$$\begin{aligned} z(x) &= z(x)(f(x) + g(x)) \\ &= z(x)f(x) + z(x)g(x) \\ &\subseteq f(x)R_\lambda + g(x)R_\lambda. \end{aligned}$$

Thus,

$$R_\lambda = f(x)R_\lambda + g(x)R_\lambda.$$

In addition, if $f(x)z_1(x) + g(x)z_2(x) = 0$,where, $z_1(x), z_2(x) \in R_\lambda$. then by (3.1), we have

$$\begin{cases} f(x)z_1(x) = f(x)[f(x)z_1(x) + g(x)z_2(x)] = 0, \\ g(x)z_2(x) = g(x)[f(x)z_1(x) + g(x)z_2(x)] = 0. \end{cases}$$

Hence, $R_\lambda = f(x)R_\lambda \oplus g(x)R_\lambda$.

2. Now, we will prove ϕ_f is a ring isomorphism. If $f(x)z(x) = 0$ in $f(x)R_\lambda$ then there exists $b(x) \in R_{u,v}[x]$ such that $f(x)z(x) = b(x)(x^{4p^s} - \lambda)$, i.e.,
$$(1 - g(x))z(x) = b(x)(x^{4p^s} - \lambda) \in R_{u,v}[x].$$
Since $g(x)$ and $(x^{4p^s} - \lambda)$ are both divisible by $(x^{2p^s} - \alpha)$, it follows that $z(x) = 0$ in $\dfrac{R_{u,v}[x]}{\langle x^{2p^s} - \alpha\rangle}$. The same holds for ϕ_g.
Thus, we have shown that ϕ_f and ϕ_g are injective, and surjective is obvious.

□

Theorem 3.1. *A subset C of R_λ is a λ-constacyclic code of length $4p^s$ over $R_{u,v}$ if and only if $C = f(x)C_1 \oplus g(x)C_2$,where C_1 is a α-constacyclic code of length $2p^s$ over $R_{u,v}$ and C_2 is a $(-\alpha)$-constacyclic code of length $2p^s$ over $R_{u,v}$. Moreover, $|C|=|C_1||C_2|$.*

Proof. By Lemma 3.1 we have $C = I \oplus J$, where I is ideal of $f(x)R_\lambda$ and J is ideal of $g(x)R_\lambda$. Since ϕ_f and ϕ_g are ring isomorphisms
$$I = f(x)C_1 \text{ and } J = g(x)C_2,$$
where C_1 is ideal of $\dfrac{R_{u,v}[x]}{\langle x^{2p^s} - \alpha\rangle}$, and C_2 is ideal of $\dfrac{R_{u,v}[x]}{\langle x^{2p^s} + \alpha\rangle}$, then we have
$$|C|=|I||J| =|C_1||C_2|.$$

□

Theorem 3.2. *Consider the λ-constacyclic code $C = f(x)C_1 \oplus g(x)C_2$ of length $4p^s$ over $R_{u,v}$ where C_1 is a α-constacyclic code of length $2p^s$ over $R_{u,v}$,and C_2 is a $(-\alpha)$-constacyclic code of length $2p^s$ over $R_{u,v}$. Then*
$$C^\perp = f(x)C_1^\perp \oplus g(x)C_2^\perp.$$

Proof. we have

$$\begin{aligned}
t(x) &= f(x)t_1(x) \oplus g(x)t_2(x) \in A(C)\\
&\Leftrightarrow \forall z_1(x) \in C_1, \forall z_2 \in C_2, [(f(x)t_1(x) \oplus g(x)t_2(x)).(f(x)z_1(x) \oplus g(x)z_2(x))] = 0\\
&\Leftrightarrow \forall z_1(x) \in C_1, z_2(x) \in C_2, f(x)t_1(x)z_1(x) = 0 \textit{ and } g(x)t_2(x)z_2(x) = 0\\
&\Leftrightarrow \forall z_1(x) \in C_1, t_1(x)z_1(x) = 0 \textit{ in } \frac{R_{u,v}[x]}{\langle x^{2p^s} - \alpha\rangle}, \forall z_2(x) \in C_2, t_2(x)z_2(x)\\
&= 0 \textit{ in } \frac{R_{u,v}[x]}{\langle x^{2p^s} + \alpha\rangle}\\
&\Leftrightarrow t_1(x) \in A(C_1) \textit{ and } t_2(x) \in A(C_2).
\end{aligned}$$

Therefore, $A(C) = f(x)A(C_1) \oplus g(x)A(C_2)$. Thus, by Proposition 2.4
$$C^\perp = f^*(x)C_1^\perp \oplus g^*(x)C_2^\perp.$$
Since $f^*(x)$ is associated with $f(x)$ and $g^*(x)$ is associated with $g(x)$, we have,
$$C^\perp = f(x)C_1^\perp \oplus g(x)C_2^\perp.$$

□

By Theorems 3.1 and 3.2, the structure of λ-constacyclic codes of length $4p^s$ over $R_{u,v}$ with the cardinality of each code, is entirely determined by the α- constacyclic code and $(-\alpha)$- constacyclic codes of length $2p^s$ over $R_{u,v}$, which are studied in detail in [26].

4. The unit λ is not a square in $R_{u,v}$

In this section, we explore the structures of all λ- constacyclic codes of length $4p^s$ over $R_{u,v}$ when the unite λ is not a square and is of the form $\lambda = \alpha + \beta u + \gamma v + \delta uv$ for non zero elements $\beta, \delta \in F_{p^m}$ and, $\alpha, \gamma \in F^*_{p^m}$ by Proposition 2.5 we note that $\alpha + \gamma v$ is not a square in K_v. We start with an important observation.

The following theorem presents some known results for $(\alpha + \gamma v)$- constacyclic codes of length $2p^s$ over the ring K_v with ideals of $S_{\alpha+\gamma v}$ which are important in this section.

Theorem 4.1. *[27]*

1. *In the ring $S_{\alpha+\gamma v}$, $(x^4-\alpha_0)^{p^s} = \gamma v$. In particular, $x^4-\alpha_0$ is nilpotent with nilpotency index of $2p^s$. This follows from the fact that*
$$S_{\alpha,\gamma} = (x^4-\alpha_0)^{p^s} = x^{4p^s} - \alpha_0^{p^s} = (\alpha+\gamma v) - \alpha = \gamma v.$$
2. *The ring $S_{\alpha+\gamma v}$ forms a chain ring with maximal ideal $\langle x^4-\alpha_0\rangle$, whose ideals are*
$$S_{\alpha+\gamma v} \supsetneq \langle x^4-\alpha_0\rangle \supsetneq ... \supsetneq \langle (x^4-\alpha_0)^{2p^s-1}\rangle \supsetneq \langle (x^4-\alpha_0)^{2p^s}\rangle = \langle 0\rangle.$$
3. *$(\alpha+\gamma v)$- constacyclic codes of length $4p^s$ over $S_{\alpha,\gamma}$ are precisely the ideals*
$$(x^4-\alpha_0)^i \subseteq S_{\alpha,\gamma},$$
where $0 \le i \le 2p^s$. The following holds:
$$|\langle (x^4-\alpha_0)^i\rangle| = p^{4m(2p^s-i)}. \tag{4.1}$$

Lemma 4.1. *1. $\forall i \in 0, ..., 2p^s-1$, then $(x^4-\alpha_0)^i \not\subset \langle u\rangle$.*

2. $(x^4-\alpha_0)^{2p^s} = 2\beta u(x^4-\alpha_0)^{p^s}$.

Proof.

1. In $S_{\alpha+\gamma v}$ by Theorem 4.1 we have the following:
$$\begin{aligned}\phi((x^4-\alpha_0)^{2p^s-1}) =& \phi(x^4-\alpha_0)^{2p^s-1} = (x^4-\alpha_0)^{2p^s-1}\\ =&(x^4-\alpha_0)^{p^s}(x^4-\alpha_0)^{p^s-1} = \gamma v(x^4-\alpha_0)^{p^s-1}.\end{aligned}$$
Thus, If $(x^4-\alpha_0)^{2p^s-1} \subseteq \langle u\rangle$, then $\phi((x^4-\alpha_0)^{2p^s-1}) = 0$.
Consequently, $\gamma v(x^4-\alpha_0)^{p^s-1} = 0$ in $S_{\alpha+\gamma v}$, leading to a contradiction.
2. In R_λ, for all $1 \le i \le p^s-1$ we have $p|\binom{p^s}{i}$. Then,
$$\begin{aligned}(x^4-\alpha_0)^{p^s} =& x^{4p^s} - \alpha_0^{p^s} + \sum_{i=1}^{p^s-1}\binom{p^s}{i}x^{4i}(-\alpha_0)^{p^s-i} = \alpha+\beta u+\gamma v+\delta uv - \alpha\\ =&\beta u+\gamma v+\delta uv.\end{aligned}$$
So
$$\begin{aligned}(x^4-\alpha_0)^{2p^s} - 2\beta u(x^4-\alpha_0)^{p^s} =& ((x^4-\alpha_0)^{p^s} - \beta u)^2\\ =&(\beta u+\gamma v+\delta uv-\beta u)^2 = (\gamma v+\delta uv)^2 = 0.\end{aligned}$$
Then
$$(x^4-\alpha_0)^{2p^s} = 2\beta u(x^4-\alpha_0)^{p^s}.$$

□

For an $(\alpha+\gamma v)$- constacyclic code and $C=\langle (x^4-\alpha_0)^i\rangle \subseteq S_{\alpha,\gamma}$ of length $4p^s$ over $S_{\alpha,\gamma}$, by Proposition 2.3 its dual $C^{\perp}$ is an $(\alpha+\gamma v)^{-1}$ - constacyclic code of length $4p^s$ over $S_{\alpha,\gamma}$. Since $(\alpha+\gamma v)(\alpha-\gamma v)=\alpha^2$,
then

$$(\alpha+\gamma v)^{-1}=(\alpha-\gamma v)\alpha^{-2}=\alpha^{-1}-\gamma v\alpha^{-2},$$

i.e.

$$C^{\perp}\subseteq S_{\alpha^{-1},\alpha^{-2},\gamma}=\frac{S_{\alpha,\gamma}[x]}{\langle x^{4p^s}-(\alpha^{-1}-\gamma v\alpha^{-2})\rangle},$$

as

$$\alpha_0^{p^s}=\alpha,\qquad (\alpha_0^{-1})^{p^s}=\alpha^{-p^s}=\alpha^{-1}.$$

Therefore, our arguments above for $S_{\alpha,\gamma}$ work the same way for $S_{\alpha^{-1},\alpha^{-2},\gamma}$.

Theorem 4.2. *The ring $S_{\alpha^{-1},-\alpha^{-2}\gamma}$ forms a chain ring with maximal ideal $\langle x^4-\alpha_0^{-1}\rangle$, whose ideals are*

$$S_{\alpha^{-1},-\alpha^{-2}\gamma}=\langle 1\rangle \supsetneq \langle x^4-\alpha_0^{-1}\rangle \supsetneq ... \supsetneq \langle (x^4-\alpha_0^{-1})^{2p^s-1}\rangle \supsetneq \langle (x^4-\alpha_0^{-1})^{2p^s}\rangle =<0> .$$

$(\alpha^{-1}-\alpha^{-2}v\gamma)$- constacyclic codes of length $4p^s$ over $S_{\alpha,\gamma}$ are precisely the ideals

$$\langle (x^4-\alpha_0^{-1})^i\rangle \subseteq S_{\alpha^{-1},-\alpha^{-2}\gamma},$$

where $0\le i\le 2p^s$.

All $(\alpha^{-1}-\alpha^{-2}v\gamma)$-constacyclic code $\langle (x^4-\alpha_0^{-1})^i\rangle \subseteq S_{\alpha^{-1},-\alpha^{-2}\gamma}$ has $p^{4m(2p^s-1)}$ codewords. We can now characterize the duals of $(\alpha+\gamma v)$-constacyclic codes.

Corollary 4.1. *$(\alpha^{-1}-v\gamma\alpha^{-2})$-constacyclic code,*

$$C^{\perp}=\langle (x^4-\alpha_0^{-1})^{2p^s-i}\rangle \subseteq S_{\alpha^{-1},\alpha^{-2},\gamma}.$$

Proof. Let $C=\langle (x^4-\alpha_0)^i\rangle \subseteq S_{\alpha,\gamma}$ be an $(\alpha+\gamma v)$- constacyclic code of the length $4p^s$ over $S_{\alpha,\gamma}$ by the theorem 4.1 we have $\mid C\mid = P^{4m(2p^s-i)}$ from Proposition 2.2

$$\mid C^{\perp}\mid = \frac{\mid R\mid^{4p^s}}{\mid C\mid}=\frac{p^{8mp^s}}{p^{4m(2p^s-i)}}=p^{4mi}, \tag{4.2}$$

the light of theorem 4.2, $C^{\perp}=<(x^4-\alpha_0^{-1})^{2p^s-i}>\subseteq S_{\alpha^{-1},\alpha^{-2}\gamma}$. □

Corollary 4.2. *The ideal $\langle v\rangle$ is the unique self-dual $(\alpha+\gamma v)$-constacyclic code of length $4p^s$ over $S_{\alpha,\gamma}$.*

Proof. We know that the ideal $\langle v\rangle$ is a self-dual $(\alpha+\gamma v)$-constacyclic code on the other hand, if $C=\langle (x^4-\alpha_0)^i\rangle \subseteq S_{\alpha,\gamma}$ is a self -dual $(\alpha+\gamma v)$- constacyclic code of length $4p^s$ over $S_{\alpha,\gamma}$.

Then since $\mid C\mid = p^{4m(2p^s-i)}$ and $\mid C^{\perp}\mid = p^{4mi}$, so $2p^s-i=i \Longrightarrow p^s=i$.

By theorem 4.1, therefore $(x^4-\alpha_0)^{p^s}=\gamma v$ i.e $\langle (x^4-\alpha_0)^{p^s}\rangle=\langle v\rangle$. □

Lemma 4.2. *Let $z(x)\in R_\lambda$ we can express it uniquely as*

$$\begin{aligned} z(x)=&\sum_{k=0}^{2p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k\\ &+u\sum_{k=0}^{2p^s-1}(a'_{0k}x^3+b'_{0k}x^2+c'_{0k}x+d'_{0k})(x^4-\alpha_0)^k. \end{aligned} \tag{4.3}$$

Proof. In $S_{\alpha+\gamma v}$, $\phi(z(x))$ can be viewed as a polynomial over K_v of degree up to $4p^s-1$ and so $\phi(z(x)) = \phi(z_1(x)) + v\phi(z_2(x))$, where $\phi(z_1(x)), \phi(z_2(x))$ are polynomials of degree up to $4p^s-1$, thus $\phi(z(x))$ can be uniquely expressed as:

$$\begin{aligned}\phi(z(x)) &= \sum_{k=0}^{p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k\\ &\quad+\gamma^{-1}(x^4-\alpha_0)^{p^s}\sum_{k=0}^{p^s-1}(a'_{0k}x^3+b'_{0k}x^2+c'_{0k}x+d'_{0k})(x^4-\alpha_0)^k\\ &= \sum_{k=0}^{p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k\\ &\quad+v\sum_{k=0}^{p^s-1}(a'_{0k}x^3+b'_{0k}x^2+c'_{0k}x+d'_{0k})(x^4-\alpha_0)^k\\ &= \sum_{k=0}^{2p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k.\end{aligned}$$

For any $p^s \le k \le 2p^s-1$ and

$$a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k} = \gamma^{-1}(a'_{0,k-p^s}x^3+b'_{0,k-p^s}x^2+c'_{0,k-p^s}x+d'_{0,k-p^s}).$$

Then in R_λ

$$z(x) = \textstyle\sum_{k=0}^{2p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k + uy(x),$$

where $y(x) \in R_\lambda$. Applying a similar process to $y(x)$, we obtain

$$y(x) = \textstyle\sum_{k=0}^{2p^s-1}(a'_{0k}x^3+b'_{0k}x^2+c'_{0k}x+d'_{0k})(x^4-\alpha_0)^k + uw(x).$$

Thus, in R_λ, we have

$$\begin{aligned}z(x) &= \sum_{k=0}^{2p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k\\ &\quad+u\sum_{k=0}^{2p^s-1}(a'_{0k}x^3+b'_{0k}x^2+c'_{0k}x+d'_{0k})(x^4-\alpha_0)^k.\end{aligned}$$

□

Proposition 4.1. *Any nonzero polynomial of degree less than 4 in $F_{p^m}[x]$ is invertible in R_λ.*

Proof. let $z(x) = ax^3+bx^2+cx+d$ be a non zero polynomial in $F_{p^m}[x]$, where $a,b,c,d \in F_{p^m}$, we need to show $z(x)$ is invertible in R_λ. If $a=b=c=0$, then $ax^3+bx^2+cx+d = d \in F^*_{p^m}$ is invertible. We consider three cases where $deg(z) = 1, 2$ and 3.
Case 1: $deg(z) = 1$ i.e $a=b=0$ then $ax^3+bx^2+cx+d = cx+d \in F^*_{p^m}$. suppose $c \neq 0$ in the ring R_λ, we have

$$(x+c^{-1}d)^{p^s}(x-c^{-1}d)^{p^s}(x^2+(c^{-1}d)^2)^{p^s}$$

$$\begin{aligned}
&=(x^4-(c^{-1}d)^4)^{p^s}=x^{4p^s}-(c^{-1}d)^{4p^s}\\
&=(\alpha+\beta u+\gamma v+\delta uv)-(c^{-1}d)^{4p^s}\\
&=(\alpha-(c^{-1}d)^{4p^s})+\beta u+\gamma v+\delta uv.
\end{aligned}$$

Since α is not square (by Proposition 2.5) we have $\alpha \neq (c^{-1}d)^{4p^s}$.

Thus, $\alpha-(c^{-1}d)^{4p^s} \in F^*_{p^m}$ is invertible, on the other hand. Since $\beta u+\gamma v+\delta uv$ is nilpotent, the expression $(x+c^{-1}d)^{p^s}(x-c^{-1}d)^{p^s}(x^2+(c^{-1}d)^2)^{p^s}$ is invertible consequently $cx+d$ also invertible.

Case 2: $deg(z)=2$ i.e $a=0, b\neq 0$ then $z(x)=ax^3+bx^2+cx+d=bx^2+cx+d \in F^*_{p^m}[x]$ Suppose $b\neq 0$ in the ring R_λ,$z(x)$ being invertible means:

$$\begin{aligned}
z(x)^{-1}=&(bx^2+cx+d)^{-1}\\
=&b^{-1}(x^2+b^{-1}cx+b^{-1}d)^{-1}\\
=&b^{-1}(x^2+c_2x+d_2)^{-1},\ where\ c_2=b^{-1}c, d_2=b^{-1}d.\\
=&b^{-1}(x^2+cx+d_2)^{p^2-1}(x^2+c_2x+d_2)^{-p^s}\\
&\times(x^2-c_2x-d_2+c_2^2)^{-p^s}(x^2-c_2x-d_2+c_2^2)^{p^s}\\
=&b^{-1}(x^2+c_2x+d_2)^{p^s-1}(x^2-c_2x-d_2+c_2^2)^{p^s}\\
&\times[(x^2+c_2x+d_2)(x^2-c_2x-d_2+c_2^2)]^{-p^s}\\
=&b^{-1}(x^2+c_2x+d_2)^{p^s-1}(x^2-c_2x-d_2+c_2^2)^{p^s}\\
&\times[x^4+(c_2^3-2c_2d_2)x+(c_2^2d_2-d_2^2)]^{-p^s}\\
=&b^{-1}(x^2+c_2x+d_2)^{p^s-1}(x^2-c_2x-d_2+c_2^2)^{p^s}\\
&\times[x^{4p^s}+(c_2^3-2c_2d_2)^{p^s}x^{p^s}++(c_2^2d_2-d_2^2)^{p^s}]^{-1}\\
=&b^{-1}(x^2+c_2x+d_2)^{p^s-1}(x^2-c_2x-d_2+c_2^2)^{p^s}\\
&\times[\alpha+\beta u+\gamma v+\delta uv+(c_2^3-2c_2d_2)^{p^s}x^{p^s}+(c_2^2d_2-d_2^2)^{p^s}]^{-1}\\
=&b^{-1}(x^2+c_2x+d_2)^{p^s-1}(x^2-c_2x-d_2+c_2^2)^{p^s}\\
&\times[(\alpha_0+c_2^2d_2-d_2^2+(c_2^3-2c_2d_2)x)^{p^s}+\beta u+\gamma v+\delta uv]^{-1},
\end{aligned}$$

Thus, $z(x)$ is invertible if and only if $\alpha_0+c_2^2d_2-d_2^2+(c_2^3-2c_2d_2)x$ is invertible, by case 1 equivalent to $\alpha_0+c_2^2d_2-d_2^2+(c_2^3-2c_2d_2)x\neq 0$. Now, $c_2^3-2c_2d_2=0$ if and only if $c_2=0$ or $c_2^2=2d_2$. on the other hand, $\alpha_0+c_2^2d_2-d_2^2=0$ if and only if $\alpha_0=d_2^2-c_2^2d_2 \Longrightarrow \alpha_0=d_2^2$ if $(c_2=0)$ or $\alpha_0=-d_2^2$ if $(c_2^2=2d_2)$. Since -1 is not square, neither can occur as α_0 is not a square, therefore $z(x)$ is invertible.

Case 3: $deg(z)=3$ i.e $a\neq 0$ and $z(x)=ax^3+bx^2+cx+d\in F_{p^m}[x]$.
Suppose $a\neq 0$ in the ring R_λ being invertible means:

$$\begin{aligned}
z(x)^{-1}=&(ax^3+bx^2+cx+d)^{-1}\\
=&a^{-1}(x^3+a^{-1}bx^2+a^{-1}cx+a^{-1}d)^{-1}\\
=&a^{-1}(x^3+b_3x^2+c_3x+d_3)^{-1},\ where\ b_3=a^{-1}b, c_3=a^{-1}c, d_3=a^{-1}d.\\
=&a^{-1}(x^3+b_3x^2+c_3x+d_3)^{p^s-1}\\
&\times(x^3+b_3x^2+c_3x+d_3)^{-p^s}(x-b_3)^{-p^s}(x-b_3)^{p^s}\\
=&a^{-1}(x^3+b_3x^2+c_3x+d_3)^{p^s-1}(x-b_3)^{p^s}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times [(x^3+b_3x^2+c_3x+d_3)(x-b_3)]^{-p^s}\\
&=a^{-1}(x^3+b_3x^2+c_3x+d_3)^{p^s-1}(x-b_3)^{p^s}\\
&\quad [x^4+(c_3-b_3^2)x^2+(d_3-b_3c_3)x-b_3d_3]^{-p^s}\\
&=a^{-1}(x^3+b_3x^2+c_3x+d_3)^{p^s-1}(x-b_3)^{p^s}\\
&\quad \times [x^{4p^s}+(c_3-b_3^2)^{p^s}x^{2p^s}+(d_3-b_3c_3)^{p^s}x^{p^s}-b_3^{p^s}d_3^{p^s}]^{-1}\\
&=a^{-1}(x^3+b_3x^2+c_3x+d_3)^{p^s-1}(x-b_3)^{p^s}\\
&\quad \times [\alpha+\beta u+\gamma v+\delta uv+(c_3-b_3^2)^{p^s}x^{2p^s}+(d_3-b_3c_3)^{p^s}x^{p^s}-b_3^{p^s}d_3^{p^s}]^{-1}\\
&=a^{-1}(x^3+b_3x^2+c_3x+d_3)^{p^s-1}(x-b_3)^{p^s}\\
&\quad \times [(c_3-b_3^2)x^2+(d_3-b_3c_3)x+(\alpha_0-b_3d_3)^{p^s}+\beta u+\gamma v+\delta uv]^{-1},
\end{aligned}
$$

Thus, $z(x)$ is invertible if and only if $(c_3-b_3^2)x^2+(d_3-b_3c_3)x+(\alpha_0-b_3d_3)$ is invertible. By case 2 equivalent to $(c_3-b_3^2)x^2+(d_3-b_3c_3)x+(\alpha_0-b_3d_3)\neq 0$. Now, $c_3-b_3^2=0$ if and only if $c_3=b_3^2$ and $d_3-b_3c_3=0$ if and only if $d_3=b_3c_3$ and $\alpha_0-b_3d_3=0$ if and only if $\alpha_0=b_3d_3$. Therefore,

$$\alpha_0=b_3^2c_3=b_3^2.b_3^2=b_3^4.$$

Since α_0 is not square so that is impossible. Thus, $z(x)$ is invertible. □

Theorem 4.3. *The ring R_λ is a local ring with the maximal ideal $\langle x^4-\alpha_0, u\rangle$ but is not a chain ring.*

Proof. First, it is clear that both u and $x^4-\alpha_0$ are nilpotent in R_λ because

$$(x^4-\alpha_0)^{p^s}=x^{4p^s}-\alpha_0^{p^s}=x^{4p^s}-\alpha=0.$$

Consider a polynomial $z(x)$ in R_λ by Lemma 4.2 $z(x)$ can be uniquely expressed

$$
\begin{aligned}
z(x)=&\sum_{k=0}^{2p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^k\\
&+u\sum_{k=0}^{2p^s-1}(a_{0k}^{'}x^3+b_{0k}^{'}x^2+c_{0k}^{'}x+d_{0k}^{'})(x^4-\alpha_0)^k\\
=&(a_{00}x^3+b_{00}x^2+c_{00}x+d_{00})+(x^4-\alpha_0)\\
&\sum_{k=1}^{2p^s-1}(a_{0k}x^3+b_{0k}x^2+c_{0k}x+d_{0k})(x^4-\alpha_0)^{k-1}\\
&+u\sum_{k=0}^{2p^s-1}(a_{0k}^{'}x^3+b_{0k}^{'}x^2+c_{0k}^{'}x+d_{0k}^{'})(x^4-\alpha_0)^k,
\end{aligned}
$$

where $a_{0k}, b_{0k}, c_{0k}, d_{0k}, a_{ok}^{'}, b_{0k}^{'}, c_{0k}^{'}, d_{ok}^{'}\in F_{p^m}$, by the Proposition 4.1 the polynomial $z(x)$ is non- invertible if $a_{00}x^3+b_{00}x^2+c_{00}x+d_{00}=0$, i.e. $z(x)\in\langle x^4-\alpha_0, u\rangle$, it means that $\langle x^4-\alpha_0, u\rangle$ forms the set of all non-invertible elements in R_λ.

Thus R_λ is a local ring with maximal ideal $\langle x^4-\alpha_0, u\rangle$.

Next, we will prove that $u\notin\langle x^4-\alpha_0\rangle$ in R_λ , Suppose contradiction that $u\in\langle x^4-\alpha_0\rangle$, then there exist polynomials $z_1(x)$ and $z_2(x)$ in R_λ such that

$$u=(x^4-\alpha_0)z_1(x)+z_2(x)(x^{4p^s}-(\alpha+\beta u+\gamma v+\delta uv)).$$

Substituting $x = \alpha_0$, yields

$$\begin{aligned} u &= (\alpha_0^4 - \alpha_0)z_1(\alpha_0) + z_2(\alpha_0)(\alpha_0^{4p^s} - (\alpha + \beta u + \gamma v + \delta uv)) \\ &= -z_2(\alpha_0)(\beta u + \gamma v + \delta uv), \end{aligned}$$

which leads to contradiction because u can not expressed as a linear combination of v in $R_{u,v}[x]$. Therefore,

$$u \notin \langle x^4 - \alpha_0 \rangle.$$

Furthermore, $(x^4 - \alpha_0) \notin \langle u \rangle$, because $(x^4 - \alpha_0)^2 \neq 0$ in R_λ.

Therefore, R_λ can not be a chain ring. □

Lemma 4.3. *Let C be an Ideal of the ring R_λ. Then*

$$C = \langle (x^4 - \alpha_0)^{\Psi} + uz(x) \rangle + uD, \tag{4.4}$$

where $0 \leq \Psi \leq 2p^s$, $\phi(C) = \langle (x^4 - \alpha_0)^{\Psi} \rangle$. And $z(x)$ is a polynomial such that

$$(x^4 - \alpha_0)^{\Psi} + uz(x) \in C.$$

As well as

$$D = \{a(x) \in R_\lambda \mid ua(x) \in C\}.$$

Proof. Since $\phi(C)$ is an ideal of $S_{\alpha+\gamma v}$, by Theorem 4.1 there exists an integer $0 \leq \Psi \leq 2p^s$ which $\phi(C) = \langle (x^4 - \alpha_0)^{\Psi} \rangle$ given that ϕ is surjective. we can find a polynomial $z(x) \in R_\lambda$ with

$$(x^4 - \alpha_0)^{\Psi} + uz(x) \in C.$$

For any polynomial $z_1(x) \in C$ there exists a polynomial $z_2(x) \in S_{\alpha+\gamma v}$ satisfying

$$\phi(z_1(x)) = z_2(x)(x^4 - \alpha_0)^{\Psi}$$

Let $z_3(x) \in R_\lambda$ such that $\phi(z_3(x)) = z_2(x)$.

Thus,

$$\phi(z_1(x)) = \phi(z_3(x))\phi((x^4 - \alpha_0)^{\Psi} + uz(x)).$$

Then,

$$z_1(x) - z_3(x)((x^4 - \alpha_0)^{\Psi} + uz(x)) \in ker(\phi) \cap C.$$

Therefore,

$$C \subseteq \langle (x^4 - \alpha_0)^{\Psi} + uz(x) \rangle + ker(\phi) \cap C.$$

Consequently,

$$C = \langle (x^4 - \alpha_0)^{\Psi} + uz(x) \rangle + ker(\phi) \cap C = \langle (x^4 - \alpha_0)^{\Psi} + uz(x) \rangle + uD,$$

where $D = ker(\phi) \cap C$. □

Theorem 4.4. *λ- constacyclic codes of length $4p^s$ over $R_{u,v}$ i.e., ideals of the ring R_λ can be classified into the following types:*
i) Type 1 : $\langle 0 \rangle$, $\langle 1 \rangle$ (Trivial ideals),
ii) Type 2 : $\langle u(x^4 - \alpha_0)^{\Psi} \rangle$, where $0 \leq \Psi \leq 2p^s - 1$ (principal ideals with nonmonic polynomial generators),

iii) Type 3 : $\langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle$*, where* $1 \leq \Psi \leq 2p^s-1$*,* $0 \leq \mu < T$*, and* $t(x)=0$ *or* $t(x)$ *is a unit of the form*

$$\sum_{k=0}^{2p^s-1} (a_k x^3 + b_k x^2 + c_k x + d_k)(x^4-\alpha_0)^k,$$

where $a_k, b_k, c_k, d_k \in F_{p^m}$ *and* $a_o x^3 + b_0 x^2 + c_0 x + d_0 \neq 0$ *(principal ideals with monic polynomial generators),*
iv) Type 4 : $\langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x), u(x^4-\alpha_0)^{\delta}\rangle$
where $1 \leq \Psi \leq 2p^s-1, 0 \leq \mu < \delta < T$ *and* $t(x)$ *is as type 3, here*

$$T = min\{w \mid u(x^4-\alpha_0)^w \in \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle\}.$$

Proof. The ideals of type 1 correspond to the trivial ideals, let C be non -trivial Ideal of R_λ. If $\phi(C)=0$ then by Lemma 4.3 we can write $C=uD$, where

$$D = \{a(x) \in R_\lambda \mid ua(x) \in C\}.$$

Since $\phi(D)$ is an ideal of $S_{\alpha+\gamma v}$, we have $\phi(D) = \langle (x^4-\alpha_0)^{\Psi}\rangle$ with $0 \leq \Psi \leq 2p^s-1$ by applying Lemma 4.3 again we obtain $D = \langle (x^4-\alpha_0)^{\Psi} + uz(x)\rangle + uE$ with $E = \{a(x) \in R_\lambda \mid ua(x) \in D\}$ and $z(x) \in R_\lambda$.

Hence we conclude $C = \langle u(x^4-\alpha_0)^{\Psi}\rangle$ this implies that C is of type 2.

If $\phi(C) \neq 0$, since $\phi(C)$ is a non-trivial Ideal of $S_{\alpha+\gamma v}$, thus we have

$$\phi(D) = \langle (x^4-\alpha_0)^{\Psi}\rangle,$$

with $1 \leq \Psi \leq 2p^s-1$. We applying Lemma 4.3 we can write

$$C = \langle (x^4-\alpha_0)^{\Psi} + uz(x)\rangle + uD,$$

where $z(x)$ is a polynomial of R_λ, and

$$D = \{a(x) \in R_\lambda \mid ua(x) \in C\}.$$

Since $uD \subseteq \langle u\rangle$ then $uD = \langle u(x^4-\alpha_0)^{\delta}\rangle$, where $0 \leq \delta \leq 2p^s$. Now, consider the expression of $z(x)$ by Lemma 4.2 we have

$$\begin{aligned} uz(x) &= u \sum_{k=0}^{2p^s-1} (a_{0k} x^3 + b_{0k} x^2 + c_{0k} x + d_{0k})(x^4-\alpha_0)^k \\ &= u(x^4-\alpha_0)^{\mu} t(x), \end{aligned}$$

where $t(x)=0$ or $t(x)$ is a unit the form

$$\sum_{k=0}^{2p^s-1} (a_k x^3 + b_k x^2 + c_k x + d_k)(x^4-\alpha_0)^k,$$

with $a_k, b_k, c_k, d_k \in F_{p^m}$ and $a_0 x^3 + b_0 x^2 + c_0 x + d_0 \neq 0$ is a non zero polynomial. So we can write

$$C = \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x), u(x^4-\alpha_0)^{\delta}\rangle.$$

Let

$$T = min\{w \mid u(x^4-\alpha_0)^w \in \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle\}$$

- If $\delta \geq T$ then $C = \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)$ this is implies C is of type 3.
- If $\delta < T$ then $C = \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x), u(x^4-\alpha_0)^{\delta}\rangle$ this is implies C is of type 4.

The value of T is important in the classification of λ- constacyclic codes of length $4p^s$ over $R_{u,v}$. Therefore, we propose to compute it in the theorem. □

Theorem 4.5. *Let T be defined as following:*

$$T = min\{w \mid u(x^4-\alpha_0)^w \in \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle\},$$

then

$$T = \begin{cases} \Psi & : \ z(x) = 0, \\ min\{\Psi, \epsilon\} : & z(x) \neq 0, \end{cases}$$

with

$$\epsilon = max\{i \mid (x^4-\alpha_0)^i \ divides \ (x^4-\alpha_0)^{2p^s+\mu-\Psi} t(x) + 2\beta(x^4-\alpha_0)^{p^s}\},$$

and

$$(x^4-\alpha_0)^{2p^s+\mu-\Psi} t(x) + 2\beta(x^4-\alpha_0)^{p^s} = (x^4-\alpha_0)^{\epsilon} z(x),$$

where $z(x)$ is either zero or a unit of the form

$$z(x) = \sum_{k=0}^{2p^s-1} c_k(x)(x^4-\alpha_0)^k$$

with $c_k(x)$ being linear polynomial over F_{p^m} and $c_0(x) \neq 0$.

Proof. Consider

$$u(x^4-\alpha_0)^w \in \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle.$$

This implies

$$u(x^4-\alpha_0)^w = f(x)((x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)).$$

For some

$$f(x) = \sum_{k=0}^{2p^s-1} a_1(x)(x^4-\alpha_0)^k + u\sum_{k=0}^{2p^s-1} b_1(x)(x^4-\alpha_0)^k \in R_{\lambda}.$$

Let

$$f_{00}(x) = \sum_{k=0}^{2p^s-\Psi-1} a_k(x)(x^4-\alpha_0)^k,$$

$$f_{01}(x) = \sum_{k=2p^s-\Psi}^{2p^s-1} a_k(x)(x^4-\alpha_0)^{k-2p^s+\Psi},$$

$$and \ \ f_1(x) = \sum_{k=0}^{2p^s-1} b_k(x)(x^4-\alpha_0)^k.$$

Here $a_k(x)$ and $b_k(x)$ are linear polynomials over F_{p^m}. Thus,

$$f(x) = f_{00}(x) + (x^4-\alpha_0)^{2p^s-\Psi} f_{01}(x) + uf_1(x).$$

Then

$$\begin{aligned} u(x^4-\alpha_0)^{\omega} =&(x^4-\alpha_0)^{\Psi} f_{00}(x) + (x^4-\alpha_0)^{2p^s} f_{01}(x) \\ &+ u((x^4-\alpha_0)^{\Psi} f_1(x) + (x^4-\alpha_0)^{\mu} t(x) f_{00}(x) \\ &+ (x^4-\alpha_0)^{2p^s+\mu-\Psi} t(x) f_{01}(x)). \end{aligned}$$

By Lemma 4.1

$$\begin{aligned} =&(x^4-\alpha_0)^{\Psi} f_{00}(x) + u(2\beta(x^4-\alpha_0)^{p^s} f_{01}(x) + (x^4-\alpha_0)^{\Psi} f_1(x) \\ &+ (x^4-\alpha_0)^{\mu} t(x) f_{00}(x) + (x^4-\alpha_0)^{2p^s+\mu-\Psi} t(x) f_{01}(x)) \end{aligned}$$

The final equality follows from applying Lemma 4.1 therefore,

$$(x^4-\alpha_0)^{\Psi} f_{00}(x) = (x^4-\alpha_0)^{\Psi} \sum_{k=0}^{2p^s-\Psi-1} a_1(x)(x^4-\alpha_0)^k \in \langle u\rangle.$$

By applying Lemma 4.1 again it follows that $f_{00}(x) = 0$ thus,

$$\begin{aligned} u(x^4-\alpha_0)^{\omega} =&u(2\beta(x^4-\alpha_0)^{p^s} f_{01}(x) \\ &+ (x^4-\alpha_0)^{\Psi} f_1(x) + (x^4-\alpha_0)^{2p^s+\mu-\Psi} t(x) f_{01}(x)) \\ =&u((x^4-\alpha_0)^{\Psi} f_1(x) + (x^4-\alpha_0)^{\epsilon} z(x) f_{01}(x)). \end{aligned}$$

Now, we consider the following cases:

- If $z(x) = 0$ then $T \geq \Psi$ since we have

  $$u(x^4-\alpha_0)^{\Psi} = u((x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)) \in \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle,$$

  it follows that $T = \Psi$.
- If $z(x) \neq 0$, then $T \geq min\{\Psi, \epsilon\}$. Conversely, we have

  $$u(x^4-\alpha_0)^{\Psi} = u((x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)) \in \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle,$$

  and

  $$\begin{aligned} u(x^4-\alpha_0)^{\epsilon} =&z(x)^{-1}(x^4-\alpha_0)^{2p^s-\Psi}((x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)) \\ \in&\langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu} t(x)\rangle. \end{aligned}$$

  Therefore , $T = min\{\Psi, \epsilon\}$.

□

Remark : In the case where $\beta = 0$, then

$$T = \begin{cases} \Psi & : \ z(x) = 0, \\ min\{\Psi, 2p^s + \mu - \Psi\} & : \ z(x) \neq 0. \end{cases}$$

We proceed to compute the number of codewords within each λ-constacyclic code of length $4p^s$ over $R_{u,v}$ denoted as C we introduce two concepts the residue and torsion of C.

$$Res(C) = \phi(C)$$
$$Tor(C) = \{c(x) \in S_{\alpha+\gamma v} \mid uc(x) \in C\}.$$

Therefore by theorem 4.1 they are of the form $\langle (x^4 - \alpha_0)^i \rangle$ where $0 \leq i \leq 2p^s$.

On the other hand , let's consider the ring homorphism

$$T : C \longrightarrow \phi(C).$$
$$z(x) \longrightarrow \phi(z(x)).$$

We have

$$ImT = Res(C).$$
$$KerT \cong uTor(C).$$

Proposition 4.2. *Let C be a code of length $4p^s$ over $R_{u,v}$, whose residue and torsion are $Res(C)$ and $Tor(C)$ Then,*

$$\mid C \mid = \mid Tor(C) \mid . \mid Res(C) \mid . \tag{4.5}$$

Lemma 4.4. *(With the theorem 12) Let C be a λ-constacyclic code of length $4p^s$ over $R_{u,v}$ Then the expressions for $Res(C)$ and $Tor(C)$ for any ideal C in R_λ The following:*
i) If $C = \langle 0 \rangle$ Then $Tor(C) = Res(C) = \langle 0 \rangle$.
ii) If $C = \langle 1 \rangle$ Then $Tor(C)Res(C) = \langle 1 \rangle$.
iii) If $C = \langle u(x^4-\alpha_0)^\Psi \rangle$ is of type 2, then $Tor(C) = \langle (x^4-\alpha_0)^\Psi \rangle$ and $Res(C) = \langle 0 \rangle$.
iv) If $C = \langle (x^4 - \alpha_0)^\Psi + u(x^4 - \alpha_0)^\mu t(x) \rangle$ is of type 3, then $Tor(C) = \langle (x^4 - \alpha_0)^T \rangle$ and $Res(C) = \langle (x^4 - \alpha_0)^\Psi \rangle$.
v) If $C = \langle (x^4 - \alpha_0)^\Psi + u(x^4 - \alpha_0)^\mu t(x), u(x^4 - \alpha_0)^\delta \rangle$ is of type 4, then $Tor(C) = \langle (x^4 - \alpha_0)^\delta \rangle$ and $Res(C) = \langle (x^4 - \alpha_0)^\Psi \rangle$. By $Res(C)$ and $Tor(C)$ in all before case, we can calculate of all Ideals in R_λ. This can be adept by applying Equations 4.1, 4.5.

5. Dual codes of λ constacyclic codes over $R_{u,v}$

In this section, we study the dual of λ- constacyclic codes of length $4p^s$ over $R_{u,v}$ when λ is not square.

Lemma 5.1. *Let C be a λ-constacyclic code of length $4p^s$ over $R_{u,v}$ and $0 \leq a, b \leq 2p^s$ such that*

$$Res(C) = \langle (x^4 - \alpha_0)^a \rangle \text{ and } Tor(C) = \langle (x^4 - \alpha_0)^b \rangle.$$

Then

$$Res(A(C)) = \langle (x^4 - \alpha_0)^{2p^s - b} \rangle \text{ and } Tor(A(C)) = \langle (x^4 - \alpha_0)^{2p^s - a} \rangle.$$

Proof. Let $Res(A(C)) = \langle (x^4-\alpha_0)^{a'}\rangle$ and $Tor(C) = \langle (x^4-\alpha_0)^{b'}\rangle$, where $0 \leq a', b' \leq 2p^s$, then

$$u(x^4-\alpha_0)^b \in C \text{ and } u(x^4-\alpha_0)^{b'} \in A(C).$$

Moreover, there exist polynomials $g(x)$ and $f(x)$ in R_λ such that

$$(x^4-\alpha_0)^a + ug(x) \in C \text{ and } (x^4-\alpha_0)^{a'} + uf(x) \in A(C).$$

The following equations hold:

$$\begin{aligned} 0 &= u(x^4-\alpha_0)^b((x^4-\alpha_0)^{a'} + uf(x)) = u(x^4-\alpha_0)^{b+a'}, \\ 0 &= u(x^4-\alpha_0)^{b'}((x^4-\alpha_0)^a + ug(x)) = u(x^4-\alpha_0)^{b'+a}. \end{aligned}$$

By Lemma 4.1, we must have

$$a' \geq 2p^s - b \quad , \text{and } b' \geq 2p^s - a$$

According to Proposition 2.2 , $\mid C \mid\mid A(C) \mid = \mid R_\lambda^n \mid = p^{8mp^s}$.Then from equations 4.1 and 4.5, we conclude that $b' = 2p^s - a$ and $a' = 2p^s - b$. □

Lemma 5.2. *By Lemma 5.1 we have*

$$A(C) = \langle (x^4-\alpha_0)^{2p^s-b} + uf(x), u(x^4-\alpha_0)^{2p^s-a}\rangle.$$

Proof. It is clear that $u(x^4-\alpha_0)^{2p^s-a} \in A(C)$ and $(x^4-\alpha_0)^{2p^s-b} + uf(x) \in A(C)$ by Lemma 5.1. Let $c(x) \in A(C)$. Then $\phi(c(x)) \in Res(A(C))$ which implies there exists
$c_0(x) \in S_{\alpha+\gamma v}$ such that

$$\phi(c(x)) = c_0(x)(x^4-\alpha_0)^{2p^s-b} = \phi(c_0(x)((x^4-\alpha_0)^{2p^s-b} + uf(x))).$$

Thus

$$\begin{gathered} c(x) = c_0(x)((x^4-\alpha_0)^{2p^s-b} + uf(x)) \\ c(x) - c_0(x)((x^4-\alpha_0)^{2p^s-b} + uf(x)) \in ker\phi. \end{gathered}$$

This implies that

$$c(x) - c_0(x)((x^4-\alpha_0)^{2p^s-b} + uf(x)) = uc_1(x) \in A(C),$$

where $c_1(x) \in S_{\alpha+\gamma v}$, then $c_1(x) \in Tor(A(C)) = \langle (x^4-\alpha_0)^{2p^s-a}\rangle$.

Then

$$c(x) \in \langle (x^4-\alpha_0)^{2p^s-b} + uf(x), u(x^4-\alpha_0)^{2p^s-a}\rangle.$$

Therefore

$$A(C) = \langle (x^4-\alpha_0)^{2p^s-b} + uf(x), u(x^4-\alpha_0)^{2p^s-a}\rangle.$$

□

We determine the annihilator of each type of ideal in Theorem 4.4. For ideals of type 1, the annihilators are straightforward to determine.

If $C = \langle 0\rangle$ then, $A(C) = \langle 1\rangle$.
If $C = \langle 1\rangle$ then, $A(C) = \langle 0\rangle$.

For other types, we need to consider two integers a and b such that

$$Res(C) = \langle (x^4-\alpha_0)^a\rangle \text{ and } Tor(C) = \langle (x^4-\alpha_0)^b\rangle.$$

These integers are determined using Lemma 4.4. Next, we find a polynomial $f(x)$ such that

$$(x^4-\alpha_0)^{2p^s-b} + uf(x) \in A(C).$$

Finally, we obtain $A(C)$ using Lemma 5.2.

Proposition 5.1. *If $C = \langle u(x^4-\alpha_0)^{\Psi}\rangle$ is ideal of type 2, then*

$$A(C) = \langle (x^4-\alpha_0)^{2p^s-\Psi}, u\rangle.$$

Proof. It is clear that

$$(x^4-\alpha_0)^{2p^s-\Psi} \in A(C).$$

Therefore, we have

$$A(C) = \langle (x^4-\alpha_0)^{2p^s-\Psi}, u\rangle.$$

□

We recall that,

$$\epsilon = max\{i \mid (x^4-\alpha_0)^i \; divides \; (x^4-\alpha_0)^{2p^s+\mu-\Psi}t(x) + 2\beta(x^4-\alpha_0)^{p^s}\},$$

and $z(x)$ is either zero or a unit of the form

$$z(x) = \sum_{i=0}^{2p^s-1} c_i(x)(x^4-\alpha_0)^i,$$

where $c_i(x)$ are linear polynomials over F_{p^m} with $c_0(x)\neq 0$ and

$$(x^4-\alpha_0)^{2p^s+\mu-\Psi}t(x) + 2\beta(x^4-\alpha_0)^{p^s} = (x^4-\alpha_0)^{\epsilon}z(x).$$

Proposition 5.2. *If $C = \langle (x^4-\alpha_0)^{\Psi} + u(x^4-\alpha_0)^{\mu}t(x)\rangle$ is an ideal of type 3, then:*

$$A(C) = \begin{cases} \langle (x^4-\alpha_0)^{2p^s-\Psi}\rangle & : \; z(x) = 0, \\ \langle (x^4-\alpha_0)^{2p^s-\epsilon} - uz(x), u(x^4-\alpha_0)^{2p^s-\Psi}\rangle & : z(x)\neq 0 \; and \; \Psi > \epsilon, \\ \langle (x^4-\alpha_0)^{2p^s-\Psi} - u(x^4-\alpha_o)^{\epsilon-\Psi}z(x)\rangle & : z(x)\neq 0 \; and \; \Psi \leq \epsilon. \end{cases}$$

Proof. We will search for a polynomial $f(x) \in S_{\alpha+\gamma v}$ that satisfies :

$$(x^4-\alpha_0)^{2p^s-T} + uf(x) \in A(C)$$

i.e.,

$$\begin{aligned} 0 &= (x^4-\alpha_0)^{2p^s-T+\Psi} + u((x^4-\alpha_0)^{2p^s-T+\mu}t(x) + (x^4-\alpha_0)^{\Psi}f(x)) \\ &= (x^4-\alpha_0)^{\Psi-T}(x^4-\alpha_0)^{2p^s} + u((x^4-\alpha_0)^{2p^s-T+\mu}t(x) + (x^4-\alpha_0)^{\Psi}f(x)) \\ &= (x^4-\alpha_0)^{\Psi-T}(2u\beta(x^4-\alpha_0)^{p^s}) + u((x^4-\alpha_0)^{2p^s-T+\mu}t(x) + (x^4-\alpha_0)^{\Psi}f(x)) \\ &= u(2\beta(x^4-\alpha_0)^{p^s+\Psi-T} + (x^4-\alpha_0)^{2p^s-T+\mu}t(x) + (x^4-\alpha_0)^{\Psi}f(x)) \\ &= u((x^4-\alpha_0)^{\Psi-T+\epsilon}z(x) + (x^4-\alpha_0)^{\Psi}f(x)). \end{aligned} \tag{5.1}$$

We consider two cases:

- If $z(x)=0$, then $T=\Psi$ and the equation 5.1 is equivalent to $0=u(x^4-\alpha_0)^{\Psi}f(x)$. In this case, we choose $f(x)=0$, and we have

$$A(C)=\langle (x^4-\alpha_0)^{2p^s-\Psi}, u(x^4-\alpha_0)^{2p^s-\Psi}\rangle = \langle (x^4-\alpha_0)^{2p^s-\Psi}\rangle.$$

- If $z(x)\neq 0$, in this case we choose

$$f(x)=-(x^4-\alpha_0)^{\epsilon-T}z(x),$$

then

$$A(C)=\langle (x^4-\alpha_0)^{2p^s-T}-u(x^4-\alpha_0)^{\epsilon-T}z(x), u(x^4-\alpha_0)^{2p^s-\Psi}\rangle.$$

□

Proposition 5.3. *If* $C=\langle (x^4-\alpha_0)^{\Psi}+u(x^4-\alpha_0)^{\mu}t(x), u(x^4-\alpha_0)^{\delta}\rangle$*, an ideal of type 4 then :*

$$A(C)=\begin{cases} \langle (x^4-\alpha_0)^{2p^s-\delta}, u(x^4-\alpha_0)^{2p^s-\Psi}\rangle & , if z(x)=0,\\ \langle (x^4-\alpha_0)^{2p^s-\delta}-u(x^4-\alpha_0)^{\epsilon-\delta}z(x), u(x^4-\alpha_0)^{2p^s-\Psi}\rangle & , if z(x)\neq 0.\end{cases}$$

Proof. We will find a polynomial $f(x)\in S_{\alpha+\gamma v}$ such that

$$(x^4-\alpha_0)^{2p^s-\delta}+uf(x)\in A(C).$$

This can be expressed as follows:

$$\begin{aligned}0&=(x^4-\alpha_0)^{2p^s-\delta+\Psi}+u(x^4-\alpha_0)^{2p^s-\delta+\mu}t(x)+(x^4-\alpha_0)^{\Psi}f(x))\\ &=(x^4-\alpha_0)^{\Psi-\delta}(x^4-\alpha_0)^{2p^s}+u((x^4-\alpha_0)^{2p^s-\delta+\mu}t(x)+(x^4-\alpha_0)^{\Psi}f(x))\\ &=(x^4-\alpha_0)^{\Psi-\delta}(2u\beta(x^4-\alpha_0)^{p^s})+u((x^4-\alpha_0)^{2p^s-\delta+\mu}t(x)+(x^4-\alpha_0)^{\Psi}f(x))\\ &=u(2\beta(x^4-\alpha_0)^{p^s+\Psi-\delta}+(x^4-\alpha_0)^{2p^s-\delta+\mu}t(x)+(x^4-\alpha_0)^{\Psi}f(x))\\ &=u((x^4-\alpha_0)^{\Psi-\delta+\epsilon}z(x)+(x^4-\alpha_0)^{\Psi}f(x)).\end{aligned}$$

Then, we choose $f(x)=-(x^4-\alpha_0)^{\epsilon-\delta}z(x)$ if $z(x)\neq 0$ and $f(x)=0$ if $z(x)=0$ resulting in:

- If $z(x)=0$, $A(C)=\langle (x^4-\alpha_0)^{2p^s-\delta}, u(x^4-\alpha_0)^{2p^s-\Psi}\rangle$.
- If $z(x)\neq 0$, $A(C)=\langle (x^4-\alpha_0)^{2p^s-\delta}-u(x^4-\alpha_0)^{\epsilon-\delta}z(x), u(x^4-\alpha_0)^{2p^s-\Psi}\rangle$.

□

6. Conclusion

This work provides a complete classification of λ-constacyclic codes of length $4p^s$ over the finite non-chain ring R_{u^2,v^2,p^m}. Distinctions between square and non-square units of λ were established, and for each case, the code structure, number of codewords, and dual codes were explicitly determined. These results offer new insights into the algebraic properties of constacyclic codes over non-chain finite rings and may serve as a foundation for further studies in this area.

Future work

Future research may explore σ-self-dual and σ-self-orthogonal codes over R_{u^2,v^2,p^m}, analyze weight distributions and minimum distances, and extend the investigation to composite code lengths or other classes of non-chain rings. Moreover, developing computational implementations in SageMath for explicit constructions and decoding algorithms represents a promising direction.

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