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Stability Analysis of a Three-Dimensional Discrete Topp Model

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Abstract Mathematical models of glucose, insulin, and pancreatic beta cell mass dynamics are essential for understanding the physiological basis of type 2 diabetes. This paper investigates the discrete-time dynamics of the Topp model to represent these interactions. We perform a comprehensive analysis of the system's trajectory, examining both local and global behavior. First, we establish the invariance of the positive trajectory and analyze the existence of fixed points. Then, we conduct a complete stability analysis, determining the local and global asymptotic stability of these fixed points. Finally, numerical examples validate the effectiveness and applicability of our theoretical findings. Additionally, we provide biological interpretations of our results.

Keywords Fixed point, periodic point, local stability, global behavior, regular operator

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1. Introduction

To sustain proper human body function, it is crucial to regulate blood glucose levels within the range of $70 - 100 mg/dl$. Insulin, secreted by beta cells in the pancreas, aids in the uptake of glucose by cells and is vital for maintaining blood sugar balance. Elevated blood glucose levels stimulate insulin secretion, which assists in reducing the concentration to a healthy range. As glucose levels drop, insulin secretion progressively diminishes. This mechanism, involving both insulin and glucagon, is essential for maintaining glucose equilibrium and preventing complications associated with diabetes [14, 26, 28].

Mathematical representations of glucose, insulin, and pancreatic beta cell mass dynamics are essential for comprehending the physiological mechanism underlying the onset of type 2 diabetes. Traditionally, type 2 diabetes was thought to arise from insulin insufficiency. The mathematical model introduced by Topp and collaborators

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is significant for examining its progression. Topp's model serves as a cornerstone for investigating the advancement of diabetes [26]. The Topp model is

$$\begin{cases} \frac{dx}{dt} = g_0 - g_1 x - cxy, \\ \frac{dy}{dt} = \frac{s_1 x^2}{s_2 + x^2} z - ky, \\ \frac{dz}{dt} = (-d_0 + r_1 x - r_2 x^2) z. \end{cases} \tag{1.1}$$

In this model, x, y, and z represent the plasma glucose concentration, insulin concentration, and the mass of functional β-cells (capable of appropriate insulin production and secretion) at time t, respectively. g_0 represents the average rate of glucose infusion per day, including hepatic glucose production, primarily from meal ingestion. $g_1 x$ represents insulin-independent glucose uptake, mainly by brain and nerve cells, while cxy depicts insulin-dependent glucose uptake, primarily by fat and muscle cells. The coefficient c represents insulin sensitivity. Insulin secretion from β-cells is assumed to follow a Hill function with a coefficient of 2, triggered by increased glucose levels, where s_1 represents the secretory capacity per β-cell. The insulin clearance rate is denoted by k. The functional β-cell mass is hypothesized to respond to glucose in a parabolic manner: moderate glucose levels promote growth, while high glucose levels exacerbate apoptosis, leading to a decrease in functional β-cell mass. d_0 represents the death rate at zero glucose, and r_1, r_2 are constants [26, 28].

In recent years, the mathematical modeling of the glucose-insulin regulatory system has been extensively studied. As indicated by recent reviews, a large number of authors are using mathematical modeling as pragmatic and theoretical tools to allow scientific understanding and efficient management of diabetes in all its aspects (Al Ali et al., 2025 [1]; Boutayeb and Lamlili, 2025 [4]; Boutayeb et al., 2025 [5]; Lamlili et al., 2025 [16]; Nasir and Mat Daud, 2022 [18]).

Li and Kuang (2007) analyzed the glucose-insulin regulatory system using a delay-based model, investigating its dynamic properties [17]. Al-Hussein et al. (2020) introduced a new time-delay model that captures chaotic behaviors in the system [2]. Subsequently, Al-Hussein, Rahma, and Jafari (2020) further examined Hopf bifurcation and chaos in this model, providing insights into its complex dynamics [3].

Fernández-Carreón et al. (2021) analyzed a fractional-order version of a mathematical model of the glucose-insulin regulatory system [13]. Rao et al. (2023) extended this analysis by considering the effects of the insulin-degrading enzyme and multiple delays, offering a more comprehensive understanding of the system's behavior [19]. Additionally, Zhao et al. (2023) developed a dynamic model based on experimental observations in mice to explore the impact of glucagon-like peptide on glucose-insulin interactions [29].

The original Topp model [26] describes the glucose, insulin, β-cell feedback system using a continuous-time framework with three nonlinear ordinary differential equations. This model revealed the existence of two stable fixed points representing physiological (healthy) and pathological (diabetic) states, and has been widely used to investigate the progression of diabetes through bifurcation analysis and numerical simulations.

Following this foundational work, several studies have investigated extensions of the Topp model, including parameter sensitivity, bifurcation behavior, and the influence of exogenous insulin. However, these studies remain confined to the continuous-time setting and largely depend on numerical simulations.

While continuous models offer theoretical richness, they may not always align with real-world data, which is often collected at discrete time intervals (e.g., daily blood glucose or insulin measurements). Nevertheless, the analysis in prior work has been largely limited to continuous-time models and mostly based on numerical methods.

Despite the growing interest in discrete dynamical systems across biological modeling, the discrete analogue of the Topp model has remained relatively unexplored. In particular, there is a lack of rigorous mathematical analysis of discrete-time models that incorporate all three physiological variables.

In contrast, our work focuses on a discrete-time formulation of the Topp model, which offers new theoretical perspectives and is more suitable for modeling real-world processes such as daily glucose measurements or insulin injections administered at discrete time intervals. Despite the biological relevance of discrete-time models, there has been limited research in this direction, and to the best of our knowledge, no rigorous mathematical analysis has been conducted on the discrete analogue of the Topp model involving all three key variables (glucose, insulin, and β-cell mass).

In this study (similar to [7–11], [22–25]), we examine the discrete-time dynamical systems related to the system (1.1) using Euler forward discretization technique. Define the operator $W : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ by

$$\begin{cases} x' = g_0 - g_1 x - cxy + x, \\ y' = \frac{s_1 x^2}{s_2 + x^2} z - ky + y, \\ z' = \left(-d_0 + r_1 x - r_2 x^2\right) z + z, \end{cases} \tag{1.2}$$

where all parameters in the model are positive. In the system (1.2), x', y', z' represent the next values corresponding to the initial state values x, y, z respectively.

The main problem for a given operator W and arbitrary initial point $\mathbf{u}^{(0)} \in \mathbb{R}^3_+$ is to describe the limit points of the trajectory $\{\mathbf{u}^{(n)}\}_{n=0}^{\infty}$, where

$$\mathbf{u}^{(n)} = W(\mathbf{u}^{n-1}) = W^n(\mathbf{u}^{(0)}).$$

Nonlinear discrete-time dynamical systems, though seemingly straightforward in structure, pose difficulties in performing a thorough analysis of operator dynamics due to the absence of a universal theory. Each operator demands a distinct research methodology. Despite substantial scientific investigations into nonlinear operator dynamical systems thus far, it is important to highlight that a full characterization of the set of limit points of trajectories for numerous dynamical systems driven by nonlinear operators remains unaccomplished [15, 21].

2. Analysis of local stability around fixed points

Let

$$\mathbb{R}^3_+ = \{(x, y, z) : x, y, z \in \mathbb{R}, x \geq 0, y \geq 0, z \geq 0\}$$

be the positive orthant of the three-dimensional space. Although the operator W is well defined in $\mathbb{R}^3$, we confine its domain to the positive orthant $\mathbb{R}^3_+$ to model the biological context of glucose, insulin, and pancreatic beta cell mass. This ensures that the dynamical system generated by W remains within the biologically meaningful region. Consequently, we select parameters for the operator W to ensure that it maps $\mathbb{R}^3_+$ to itself. We adopt the following notation

$$A = \frac{r_1 + \sqrt{r_1^2 + 4r_2(1-d_0)}}{2r_2},\ B = \frac{1-g_1}{c},\ C = \frac{(1-g_1)k}{s_1 c}.$$

Lemma 2.1. *If*

$$0 < g_1 < 1,\ 0 < k \le 1,\ 0 < d_0 \le 1,\ g_0/g_1 \le A,\ r_1^2 \le 4r_2 d_0 \tag{2.1}$$

then the operator (1.2) maps the set

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3_+ : g_0 \le x \le A, 0 \le y \le B, 0 \le z \le C\right\},$$

to itself.

Proof. We aim to demonstrate that for any arbitrary point (x, y, z) within the set Ω, if conditions (2.1) are met, then the point (x', y', z') must also reside within Ω. For any $(x, y, z) \in \Omega$, defined by $g_0 \le x \le A, 0 \le y \le B, 0 \le z \le C$ we have

$$\begin{aligned} x' &= g_0 + x(1 - g_1 - cy) \ge g_0 + x(1 - g_1 - c \cdot B) \\ &= g_0 + x(1 - g_1 - c \cdot \frac{1-g_1}{c}) = g_0, \\ x' &= g_0 + x(1 - g_1 - cy) \le g_0 + x(1-g_1) \le g_0 + A(1-g_1) \\ &= A - g_1(A - g_0/g_1) \le A. \end{aligned}$$

Thus $g_0 \le x' \le A$.

Since $0 < k \le 1$, it is clear that $y' \ge 0$. We will now show that $y' \le B$.

$$\begin{aligned} y' &= \frac{s_1 x^2}{s_2 + x^2} z + (1-k)y \le s_1 \cdot C + (1-k)B \\ &= s_1 \cdot \frac{(1-g_1)k}{s_1 c} + (1-k)B = kB + (1-k)B = B. \end{aligned}$$

It follows that $0 \le y' \le B$.

It can be seen that the number A is a positive solution to the equation $1 - d_0 + r_1 x - r_2 x^2 = 0$. When x ranges from 0 to A, the expression $1 - d_0 + r_1 x - r_2 x^2$ is always non-negative. Since $[g_0, A] \subset [0, A]$, then $z' \ge 0$.

$$\begin{aligned} z' &= \left(1 - d_0 + r_1 x - r_2 x^2\right) z \le \left(1 - d_0 + \frac{r_1^2}{2r_2} - \frac{r_1^2}{4r_2}\right) C \\ &= \left(1 - \frac{4r_2 d_0 - r_1^2}{4r_2}\right) C \le C. \end{aligned}$$

It follows that z' also changes from 0 to C. Thus $(x', y', z') \in \Omega$. □

2.1. Fixed points

We begin by investigating the existence of fixed points. A point $u \in \Omega$ is considered a fixed point of the operator W if it satisfies the condition $W(u) = u$.

The equation $W(u) = u$ is the following system

$$\begin{cases} x = g_0 + x(1 - g_1 - cy), \\ y = \frac{s_1 x^2}{s_2 + x^2} z + (1 - k)y, \\ z = \left(1 - d_0 + r_1 x - r_2 x^2\right) z. \end{cases} \tag{2.2}$$

The third equation in system (2.2) implies that either $z = 0$ or

$$r_2 x^2 - r_1 x + d_0 = 0. \tag{2.3}$$

The solutions of equation (2.3) are

$$x_{1,2} = \frac{r_1 \pm \sqrt{r_1^2 - 4r_2 d_0}}{2r_2}.$$

Thus, taking into account conditions (2.1), system (2.2) has solutions

$$x_1^* = \frac{g_0}{g_1}, y_1^* = 0, z_1^* = 0,$$
$$x_2^* = \frac{r_1}{2r_2}, y_2^* = \frac{g_0 - g_1 x_2^*}{c x_2^*}, z_2^* = \frac{k y_2^* (s_2 + x_2^{*2})}{s_1 x_2^{*2}}.$$

Now let's find additional conditions for the parameters so that the point (x_2^*, y_2^*, z_2^*) belongs to the set Ω.

If $r_1^2 < 4r_2 d_0$, equation (2.3) has no solution, i.e., $x_2^* \notin [g_0, A]$, if $r_1^2 = 4r_2 d_0$ and $2r_2 g_0 < r_1$ then $x_2^* \in [g_0, A]$. It follows that $y_2 \leq B$. Indeed,

$$y_2^* = \frac{g_0 - g_1 x_2^*}{c x_2^*} \leq \frac{x_2^* - g_1 x_2^*}{c x_2^*} = \frac{1 - g_1}{c} = B.$$

If $g_1 r_1 < 2r_2 g_0$ then $y_2^* \geq 0.$, i.e., $y_2^* \in [0, B]$. Since x_2^* and y_2^* are non-negative, it follows that z_2^* is also non-negative. We solve the inequality $z_2^* \leq C$ and form the condition $s_2 r_2 (2r_2 g_0 - g_1 r_1) \leq d_0 (r_1 - 2r_2 g_0)$ for the parameters.

So, if the parameters satisfy the following conditions

$$r_1^2 = 4r_2 d_0, \ \frac{g_1 r_1}{2r_2} < g_0 < \frac{r_1}{2r_2}, \ s_2 \leq \frac{d_0 (r_1 - 2r_2 g_0)}{r_2 (2r_2 g_0 - g_1 r_1)}, \tag{2.4}$$

then the point (x_2^*, y_2^*, z_2^*) belongs to the set Ω. So the fixed points are as follows:

$$u_1^* = \left(\frac{g_0}{g_1}, 0, 0\right), \quad u_2^* = (x^*, y^*, z^*)$$

where

$$x^* = \frac{r_1}{2r_2}, \ y^* = \frac{g_0 - g_1 x^*}{c x^*}, \ z^* = \frac{k y^* (s_2 + x^{*2})}{s_1 x^{*2}}. \tag{2.5}$$

2.2. Local behavior of fixed points

We primarily employ techniques to investigate the local behavior of fixed points within this subsection.

Let $J(u^*)$ be a Jacobian matrix and λ_i (i=1,2,3) are its eigenvalues.

Definition 2.1. (see [12])

- If $|\lambda_i| \neq 1$ then the fixed point u^* of the operator W is called hyperbolic;
- if $|\lambda_i| < 1$ then the hyperbolic fixed point u^* of the operator W is called attracting;
- if $|\lambda_i| > 1$ then the hyperbolic fixed point u^* of the operator W is called repelling;
- in all other cases, the hyperbolic fixed point u^* is called a saddle.

We use the following lemma to analyze the local behavior of fixed points.

Lemma 2.2. *(see [27], [25]) Assume λ_1 and λ_2 are two roots of $F(\lambda) = \lambda^2 + B^*\lambda + C^* = 0$, where B^* and C^* are two real constants. Assuming $F(1) > 0$ the following conditions hold:*

1) Both eigenvalues, $|\lambda_1|$ and $|\lambda_2|$ are less than 1 if and only if $F(-1) > 0$ and $C^* < 1$;

2) Both eigenvalues, $|\lambda_1|$ and $|\lambda_2|$ are greater than 1 if and only if $F(-1) > 0$ and $C^* > 1$.

First, we calculate the Jacobian matrix of the operator (1.2) at the fixed point $u = (x, y, z)$. It is as follows:

$$J(u) = \begin{pmatrix} 1 - g_1 - cy & -cx & 0 \\ \frac{2s_1s_2x}{(s_2+x^2)^2}z & 1-k & \frac{s_1x^2}{s_2+x^2} \\ (r_1 - 2r_2x)z & 0 & 1 - d_0 + r_1x - r_2x^2 \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix at the fixed point $u_1^* = \left(\frac{g_0}{g_1}, 0, 0\right)$ are as follows

$$\lambda_1 = 1 - g_1, \ \lambda_2 = 1 - k, \ \lambda_3 = 1 - d_0 + \frac{r_1g_0}{g_1} - \frac{r_2g_0^2}{g_1^2}.$$

By (2.1) we have $0 < \lambda_1 < 1, 0 \leq \lambda_2 < 1$. We write λ_3 as follows.

$$\lambda_3 = 1 - d_0 + \frac{r_1g_0}{g_1} - \frac{r_2g_0^2}{g_1^2} = 1 - r_2\left(\frac{g_0}{g_1} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2d_0}{4r_2}.$$

If we consider that $r_1^2 \leq 4r_2d_0$, then $\lambda_3 < 1$.

Let us now consider an analysis of the fixed point $u_2^* = (x^*, y^*, z^*)$. The Jacobian matrix evaluated at the fixed point u_2^* is given by

$$J(u_2^*) = \begin{pmatrix} 1 - g_1 - cy^* & -cx^* & 0 \\ \frac{2s_1s_2x^*}{(s_2+x^{*2})^2}z^* & 1-k & \frac{s_1x^{*2}}{s_2+x^{*2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

You can see that one of the eigenvalues is equal to one. The remaining eigenvalues are the roots of the equation

$$(1-g_1-cy^*-\lambda)(1-k-\lambda)+\frac{2s_1s_2x^*}{(s_2+x^{*2})^2}cx^*z^*=0.$$

We have

$$F(\lambda)=\lambda^2+B^*\lambda+C^*=0, \tag{2.6}$$

where

$$\begin{aligned}
B^* &= k+g_1+cy^*-2=k+\frac{g_0}{x^*}-2,\\
C^* &= (1-g_1-cy^*)\,(1-k)+\frac{2s_1s_2x^*}{(s_2+x^{*2})^2}cx^*z^*\\
&= \left(1-\frac{g_0}{x^*}\right)(1-k)+\frac{2cs_1s_2x^{*2}}{(s_2+x^{*2})^2}\cdot\frac{ky_2^*(s_2+x^{*2})}{s_1x^{*2}}\\
&= \left(1-\frac{g_0}{x^*}\right)(1-k)+\frac{2s_2kcy^*}{s_2+x^{*2}}.
\end{aligned}$$

From (2.6) it is clear that one of the eigenvalues is equal to one. Let us determine the sign of $F(1)$, $F(-1)$ and C^*-1 under conditions (2.1), (2.4).

$$\begin{aligned}
F(1) &= 1+B^*+C^*\\
&= 1+k+\frac{g_0}{x^*}-2+\left(1-\frac{g_0}{x^*}\right)(1-k)+\frac{2s_2kcy^*}{s_2+x^{*2}}\\
&= \frac{g_0k}{x^*}+\frac{2s_2kcy^*}{s_2+x^{*2}}>0,\\
F(-1) &= 1-B^*+C^*\\
&= 1-k-\frac{g_0}{x^*}+2+\left(1-\frac{g_0}{x^*}\right)(1-k)+\frac{2s_2kcy^*}{s_2+x^{*2}}\\
&= \left(2-\frac{g_0}{x^*}\right)(2-k)+\frac{2s_2kcy^*}{s_2+x^{*2}}>0,
\end{aligned}$$

$$\begin{aligned}
C^*-1 &= \left(1-\frac{g_0}{x^*}\right)(1-k)+\frac{2s_2kcy^*}{s_2+x^{*2}}-1\\
&= \frac{2s_2k(g_0-g_1x^*)}{x^*(s_2+x^{*2})}-\frac{g_0}{x^*}(1-k)-k\\
&= \frac{1}{x^*(s_2+x^{*2})}\cdot\left(2s_2k(g_0-g_1x^*)-g_0(1-k)(s_2+x^{*2})-kx^*(s_2+x^{*2})\right)\\
&\le \frac{k}{x^*(s_2+x^{*2})}\cdot\left(2s_2(g_0-g_1x^*)-x^*(s_2+x^{*2})\right)\\
&= \frac{k}{x^*(s_2+x^{*2})}\cdot\left(s_2(2(g_0-g_1x^*)-x^*)-x^{*3}\right)\\
&\le \frac{k}{x^*(s_2+x^{*2})}\cdot\left(\frac{x^{*2}(x^*-g_0)}{g_0-g_1x^*}\cdot(2(g_0-g_1x^*)-x^*)-x^{*2}\right)\\
&= \frac{kx^*}{(s_2+x^{*2})(g_0-g_1x^*)}\cdot\left(-(1+g_1)x^{*2}+2g_0(1+g_1)x^*-2g_0^2\right)
\end{aligned}$$

$$= -\frac{k(1+g_1)x^*}{(s_2+x^{*2})(g_0-g_1x^*)} \cdot \left((x^*-g_0)^2 + \frac{(1-g_1)g_0^2}{1+g_1}\right) < 0.$$

Lemma 2.2 1) establishes that $|\lambda_{2,3}| < 1$. Consequently, the subsequent theorem characterizes the local behavior around fixed points.

Theorem 2.1. *The local behavior of fixed points for the operator W defined by (1.2) is as follows:*

condition	*fixed points*	*local behavior*
$r_1^2 < 4r_2d_0$	u_1^*	*attracting*
$r_1^2 = 4r_2d_0$, $\frac{g_1r_1}{2r_2} < g_0 < \frac{r_1}{2r_2}$, $s_2r_2(2r_2g_0 - g_1r_1) \leq d_0(r_1 - 2r_2g_0)$	u_1^*, u_2^*	*attracting, non-hyperbolic*

Remark 2.1. Although u_2^* is a non-hyperbolic fixed point, it is semi-attractive because two of its eigenvalues have absolute values less than one.

3. Periodic points

This subsection investigates the existence of periodic points for the operator W within in the set Ω. This analysis contributes to understanding the global dynamics of the system.

Definition 3.1. (see [20]) A point $\mathbf{u} \in \Omega$ is termed periodic point of W if there exists $p \in \mathbb{N}$ so that $W^p(\mathbf{u}) = \mathbf{u}$. The smallest positive integer p that satisfies $W^p(\mathbf{u}) = \mathbf{u}$ is termed the prime period or the least period of the point $\mathbf{u}$.

Theorem 3.1. *The operator W, as defined in equation (1.2), has no periodic points of period p in the set Ω for all $p \geq 2$.*

Proof. The following system of equations provides a means to locate periodic points with a period of p.

$$\begin{cases} x = x^{(p)} = g_0 + x^{(p-1)}\left(1 - g_1 - cy^{(p-1)}\right), \\ y = y^{(p)} = \frac{s_1\left(x^{(p-1)}\right)^2}{s_2+\left(x^{(p-1)}\right)^2} z^{(p-1)} + (1-k)y^{(p-1)}, \\ z = z^{(p)} = \left(1 - d_0 + r_1x^{(p-1)} - r_2\left(x^{(p-1)}\right)^2\right)z^{(p-1)}. \end{cases} \tag{3.1}$$

From the third equation of system (3.1) we have

$$\begin{aligned} z &= z^{(p-1)}\left(1 - r_2\left(x^{(p-1)} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2d_0}{4r_2}\right) \\ &= z^{(p-2)}\left(1 - r_2\left(x^{(p-1)} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2d_0}{4r_2}\right) \\ &\cdot\left(1 - r_2\left(x^{(p-2)} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2d_0}{4r_2}\right) \\ &= \ldots = z\left(1 - r_2\left(x^{(p-1)} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2d_0}{4r_2}\right) \end{aligned}$$

$$\cdot\left(1-r_2\left(x^{(p-2)}-\frac{r_1}{2r_2}\right)^2+\frac{r_1^2-4r_2d_0}{4r_2}\right)\cdot ...$$
$$\cdot\left(1-r_2\left(x-\frac{r_1}{2r_2}\right)^2+\frac{r_1^2-4r_2d_0}{4r_2}\right).$$

Since $z\neq 0$, we have

$$\left(1-r_2\left(x^{(p-1)}-\frac{r_1}{2r_2}\right)^2+\frac{r_1^2-4r_2d_0}{4r_2}\right)$$
$$\cdot\left(1-r_2\left(x^{(p-2)}-\frac{r_1}{2r_2}\right)^2+\frac{r_1^2-4r_2d_0}{4r_2}\right)\cdot ...$$
$$\cdot\left(1-r_2\left(x-\frac{r_1}{2r_2}\right)^2+\frac{r_1^2-4r_2d_0}{4r_2}\right)=1. \tag{3.2}$$

Under the condition $r_1^2<4r_2d_0$, the left side of equation (3.2) will always be less than 1. In this case, the equation will not have roots, which means that operator (1.2) does not have a p-periodic point in the set Ω.

Let $r_1^2=4r_2d_0$. Then, for equation (3.2) to have a root, it is necessary and sufficient to satisfy the relations

$$x^{(p-1)}=x^{(p-2)}=...=x=\frac{r_1}{2r_2}.$$

These relationships hold only at a fixed point $\frac{r_1}{2r_2}$. So, in this case, operator (1.2) does not have p-periodic points in the set Ω. □

4. Global behavior

In this section, we examine the behavior of trajectories

$$\left(x^{(n)},y^{(n)},z^{(n)}\right)=W^n\left(x^{(0)},y^{(0)},z^{(0)}\right),$$

where $n\geq 1$, for any initial point $\left(x^{(0)},y^{(0)},z^{(0)}\right)\in\Omega$.

Let

$$\begin{cases} x^{(n)}=g_0+x^{(n-1)}\left(1-g_1-cy^{(n-1)}\right),\\ y^{(n)}=\frac{s_1\left(x^{(n-1)}\right)^2}{s_2+\left(x^{(n-1)}\right)^2}z^{(n-1)}+(1-k)y^{(n-1)},\\ z^{(n)}=\left(1-d_0+r_1x^{(n-1)}-r_2\left(x^{(n-1)}\right)^2\right)z^{(n-1)}.\end{cases} \tag{4.1}$$

Lemma 4.1. *Suppose that (2.1) holds. For sequences $x^{(n)}$, $y^{(n)}$ and $z^{(n)}$, the following property holds:*

- ***(i)*** *if $r_1^2<4r_2d_0$, then the sequence $z^{(n)}$ is monotonically decreasing and converges to zero;*
- ***(ii)*** *if the sequence $z^{(n)}$ converges to zero, then the sequence $y^{(n)}$ also has a limit and converges to zero;*

(iii) *if the sequence $y^{(n)}$ converges to zero, then the sequence $x^{(n)}$ also has a limit and converges to $\frac{g_0}{g_1}$.*

Proof. **(i)**. Based on the third equation within system (4.1), we obtain

$$\begin{aligned}\frac{z^{(n)}}{z^{(n-1)}} &= 1 - d_0 + r_1 x^{(n-1)} - r_2 \left(x^{(n-1)}\right)^2 \\ &= 1 - r_2 \left(x^{(n-1)} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2 d_0}{4r_2} < 1.\end{aligned}$$

This implies that $z^{(n)}$ is a decreasing sequence.

$$\begin{aligned}z^{(n)} &= \left(1 - d_0 + r_1 x^{(n-1)} - r_2 \left(x^{(n-1)}\right)^2\right) z^{(n-1)} \\ &= \left(1 - r_2 \left(x^{(n-1)} - \frac{r_1}{2r_2}\right)^2 + \frac{r_1^2 - 4r_2 d_0}{4r_2}\right) z^{(n-1)} \\ &< \left(1 + \frac{r_1^2 - 4r_2 d_0}{4r_2}\right) z^{(n-1)} < \left(1 + \frac{r_1^2 - 4r_2 d_0}{4r_2}\right)^2 z^{(n-2)} \\ &< ... < \left(1 + \frac{r_1^2 - 4r_2 d_0}{4r_2}\right)^n z^{(0)}.\end{aligned}$$

The sequence $z^{(n)}$ varies within the range

$$0 \le z^{(n)} < \left(1 + \frac{r_1^2 - 4r_2 d_0}{4r_2}\right)^n z^{(0)}.$$

Accordingly

$$\lim_{n\to\infty} z^{(n)} = 0.$$

(ii). By analyzing the second equation of (4.1) and considering the boundedness of the sequence $x^{(n)}$, we have

$$\frac{s_1 g_0^2}{s_2 + g_0^2} z^{(n-1)} \le y^{(n)} - (1-k) y^{(n-1)} \le \frac{s_1 A^2}{s_2 + A^2} z^{(n-1)}.$$

Since the sequence $z^{(n)}$ converges to zero, it follows that the sequence $y^{(n)} - (1 - k)y^{(n-1)}$ also converges to zero.

Since $y^{(n)}$ is bounded, it has a well-defined upper limit, denoted by α. Then there must exist a subsequence $y^{(n_j)}$ that converges to α. Furthermore, the fact that $y^{(n)} - (1 - k)y^{(n-1)}$ converges to zero implies another subsequence $y^{(n_j-1)}$ approaches $\frac{\alpha}{1-k}$. Since k is between 0 and 1, $\frac{\alpha}{1-k}$ must be greater than or equal to α. However, since α is an upper limit, it follows that $\frac{\alpha}{1-k} \le \alpha$. Therefore, $\frac{\alpha}{1-k}$ can only equal α, which implies $\alpha = 0$. We conclude that the sequence $y^{(n)}$ has a limit, which is equal to zero.

(iii). Using the first equation of (4.1) and the fact that $x^{(n)}$ is bounded, we can conclude that

$$g_0 - cAy^{(n-1)} \le x^{(n)} - (1 - g_1)x^{(n-1)} \le g_0 - cg_0 y^{(n-1)}. \tag{4.2}$$

Since $y^{(n)}$ converges to zero, it follows from (4.2) that

$$\lim_{n\to\infty} \left(x^{(n)} - (1 - g_1)x^{(n-1)}\right) = g_0.$$

Since the sequence $x^{(n)}$ is bounded, it admits upper and lower limits, denoted by α and β, respectively. Suppose, for the sake of contradiction, that the sequence does not converge; that is, $\alpha \neq \beta$. Then, by the definition of $\limsup$ and $\liminf$, there exist two subsequences $x^{(n_i)} \to \alpha$ and $x^{(n_j)} \to \beta$ with $\alpha > \beta$.

Using the recursive relation

$$x^{(n+1)} = (1-g_1)x^{(n)} + g_0,$$

we derive the following:

$$\begin{aligned}
&\lim_{i\to\infty}\left(x^{(n_i)} - (1-g_1)x^{(n_i-1)}\right) = g_0\\
&\Rightarrow \quad \lim_{i\to\infty} x^{(n_i-1)} = \frac{\alpha - g_0}{1-g_1} \le \alpha \quad \Rightarrow \quad \alpha \le \frac{g_0}{g_1},\\
&\lim_{i\to\infty}\left(x^{(n_i+1)} - (1-g_1)x^{(n_i)}\right) = g_0\\
&\Rightarrow \quad \lim_{i\to\infty} x^{(n_i+1)} = g_0 + (1-g_1)\alpha \le \alpha \quad \Rightarrow \quad \alpha \ge \frac{g_0}{g_1}.
\end{aligned}$$

Thus, we obtain $\alpha = \frac{g_0}{g_1}$. Similarly, for the subsequence converging to β:

$$\begin{aligned}
&\lim_{j\to\infty}\left(x^{(n_j)} - (1-g_1)x^{(n_j-1)}\right) = g_0\\
&\Rightarrow \quad \lim_{j\to\infty} x^{(n_j-1)} = \frac{\beta - g_0}{1-g_1} \ge \beta \quad \Rightarrow \quad \beta \ge \frac{g_0}{g_1},\\
&\lim_{j\to\infty}\left(x^{(n_j+1)} - (1-g_1)x^{(n_j)}\right) = g_0\\
&\Rightarrow \quad \lim_{j\to\infty} x^{(n_j+1)} = g_0 + (1-g_1)\beta \ge \beta \quad \Rightarrow \quad \beta \le \frac{g_0}{g_1}.
\end{aligned}$$

Hence, we also conclude that $\beta = \frac{g_0}{g_1}$.

This contradicts the assumption that $\alpha \neq \beta$. Therefore, the sequence $x^{(n)}$ must converge, and its limit is:

$$\lim_{n\to\infty} x^{(n)} = \frac{g_0}{g_1}.$$

□

Definition 4.1. An operator W is called regular if, for any initial point $\mathbf{u} \in \Omega$, the following limit exists:

$$\lim_{n\to\infty} W^n \mathbf{u}.$$

Theorem 4.1. *If the conditions (2.1) and $r_1^2 < 4r_2d_0$ hold, then the operator W is regular.*

Proof. For any initial point $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \in \Omega$, we have (see Lemma 4.1)

$$\lim_{n\to\infty} W^n\left(x^{(0)}, y^{(0)}, z^{(0)}\right) = \left(\frac{g_0}{g_1}, 0, 0\right).$$

For example, see Figure 1. □

To analyze the global behavior of system (1.2) under the condition $r_1^2 = 4r_2d_0$, we partition the set Ω into invariant subsets.

A set S is said to be invariant under the action of W if $W(S)$ is a subset S.

Proposition 4.1. *The sets*

$$\Omega_1 = \left\{(x, y, z) \in \Omega : x \leq \frac{g_0}{g_1}\right\},$$
$$\Omega_2 = \{(x, y, z) \in \Omega_1 : z < z^*\},$$

are invariant with respect to the operator W.

Proof. Consider an arbitrary point (x, y, z) within the set Ω_1, which is defined by the condition $g_0 \leq x \leq \frac{g_0}{g_1}$. Then

$$\begin{aligned} x' - \frac{g_0}{g_1} &= g_0 + x\,(1 - g_1 - cxy) - \frac{g_0}{g_1} \\ &\leq g_0 + \frac{g_0}{g_1}\,(1 - g_1 - cxy) - \frac{g_0}{g_1} = -\frac{g_0 cxy}{g_1} \leq 0. \end{aligned}$$

So Ω_1 is an invariant set.

Now we show that Ω_2 is an invariant set. Let $\forall(x, y, z) \in \Omega_2. \Rightarrow z < z^*$. Then

$$\begin{aligned} z' &= z\left(1 - d_0 + r_1 x - r_2 x^2\right) \\ &= z\left(1 - r_2(x - \frac{r_1}{2r_2})^2 + \frac{r_1^2 - 4r_2 d_0}{4r_2}\right) \\ &\leq z < z^*. \end{aligned}$$

Thus $(x', y', z') \in \Omega_2$, i.e., Ω_2 is an invariant set.

□

The subsequent theorem provides a complete characterization of the limit point set for the trajectory originating from any initial condition $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \in \Omega$.

Theorem 4.2. *Assume that (2.1) and (2.4) hold. For the operator W defined by (1.2) the following hold:*

- ***(i)*** *for any natural number n, if $x^{(n)} > \frac{g_0}{g_1}$, then the trajectory originating from the initial point $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \in \Omega \setminus \Omega_1$ converges to the limit point u_1^*.*
- ***(ii)*** *the trajectory of any initial point $\left(x^{(0)}, y^{(0)}, z^{(0)}\right)$ in the set Ω_2 converges to fixed point u_1^*.*
- ***(iii)*** *for any natural number n, if $z^{(n)} > z^*$, then the trajectory originating from the initial point $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \in \Omega_1 \setminus \Omega_2$ converges to the fixed point u_2^*.*

Proof. Let the trajectory for $r_1^2 = 4r_2 d_0$ be as follows.

$$\begin{cases} x^{(n)} = g_0 + x^{(n-1)}\left(1 - g_1 - cy^{(n-1)}\right), \\ y^{(n)} = \frac{s_1\left(x^{(n-1)}\right)^2}{s_2 + \left(x^{(n-1)}\right)^2} z^{(n-1)} + (1 - k)y^{(n-1)}, \\ z^{(n)} = -r_2\left(x^{(n-1)} - \frac{r_1}{2r_2}\right)^2 z^{(n-1)} + z^{(n-1)}. \end{cases} \tag{4.3}$$

(i). Let, according to the conditions of the theorem, all values of $x^{(n)}$ be greater than $\frac{g_0}{g_1}$. Then $x^{(n)}$ and $z^{(n)}$ are sequences there will be a decreasing sequence. Indeed,

$$x^{(n)} - x^{(n-1)} = g_0 - g_1 x^{(n-1)} - cx^{(n-1)}y^{(n-1)}$$

$$= g_1\left(\frac{g_0}{g_1} - x^{(n-1)}\right) - cx^{(n-1)}y^{(n-1)} \leq 0.$$

$$z^{(n)} - z^{(n-1)} = -r_2\left(x^{(n-1)} - \frac{r_1}{2r_2}\right)^2 z^{(n-1)} \leq 0.$$

Since both sequences $x^{(n)}$ and $z^{(n)}$ are decreasing and bounded from below, we have:

$$\lim_{n\to\infty} x^{(n)} \geq \frac{g_0}{g_1}, \ \lim_{n\to\infty} z^{(n)} \geq 0. \tag{4.4}$$

Our estimate for $x^{(n)}$ and $z^{(n)}$ are determined as follows:

$$\begin{aligned}
x^{(n)} &= g_0 - g_1x^{(n-1)} - cx^{(n-1)}y^{(n-1)} + x^{(n-1)} < g_0 + (1-g_1)x^{(n-1)}\\
&< g_0 + (1-g_1)\left(g_0 + (1-g_1)x^{(n-2)}\right)\\
&< g_0 + g_0(1-g_1) + ... + g_0(1-g_1)^{n-1} + (1-g_1)^n x^{(0)}\\
&= \frac{g_0}{g_1}\left(1-(1-g_1)^n\right) + (1-g_1)^n x^{(0)}.\\
z^{(n)} &= \left(1 - r_2\left(x^{(n-1)} - \frac{r_1}{2r_2}\right)^2\right) z^{(n-1)}\\
&< \left(1 - r_2\left(\frac{g_0}{g_1} - \frac{r_1}{2r_2}\right)^2\right) z^{(n-1)}\\
&< \left(1 - r_2\left(\frac{g_0}{g_1} - \frac{r_1}{2r_2}\right)^2\right)^2 z^{(n-2)} < ...\\
&< \left(1 - r_2\left(\frac{g_0}{g_1} - \frac{r_1}{2r_2}\right)^2\right)^n z^{(0)}.
\end{aligned}$$

From the above inequalities, we have

$$\lim_{n\to\infty} x^{(n)} \leq \frac{g_0}{g_1}, \ \lim_{n\to\infty} z^{(n)} \leq 0. \tag{4.5}$$

From inequalities (4.4) and (4.5), we deduce that: $x^{(n)} \to \frac{g_0}{g_1}, \ z^{(n)} \to 0$. From the first equation (4.3) it follows that $y^{(n)} \to 0$.

(ii). Let $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \in \Omega_2$. Given that Ω_2 is an invariant set, it follows that $g_0 \leq x^{(n)} \leq \frac{g_0}{g_1}$, $0 \leq y^{(n)} \leq C$, and $0 \leq z^{(n)} < z^*$ for any natural number n. Furthermore, since $z^{(n)}$ exhibits strict monotonicity (i.e., it is strictly decreasing), we have $0 \leq \lim_{n\to\infty} z^{(n)} < z^*$. Assuming that $\lim_{n\to\infty} z^{(n)} = \overline{z} \neq 0$, it becomes evident from the third equation of system (4.3) that the existence of the limit of $z^{(n)}$ necessitates the existence of the limit of $x^{(n)}$. Analogously, from the first equation of system (4.3), the existence of the limit of $x^{(n)}$ implies the existence of the limit $y^{(n)}$. We denote the limits of the sequences $x^{(n)}$ and $y^{(n)}$ by $\overline{x}$ and $\overline{y}$, respectively. Subsequently, from system (4.3), we derive the following system of equations:

$$\begin{cases} \overline{x} = g_0 + \overline{x}(1 - g_1 - c\overline{y}), \\ \overline{y} = \frac{s_1 \overline{x}^2}{s_2 + \overline{x}^2} \overline{z} + (1-k)\overline{y}, \\ \overline{z} = \left(1 - r_2 \left(\overline{x} - \frac{r_1}{2r_2}\right)^2\right) \overline{z}. \end{cases} \tag{4.6}$$

Based on the results in (4.6)

$$\overline{x} = \frac{r_1}{2r_2} = x^*, \ \overline{y} = y^*, \ \overline{z} = z^* \quad \text{(see (2.5))}.$$

This contradiction implies that $z^{(n)} \to 0$. Due to the convergence of $z^{(n)}$ to zero, Lemma 4.1 (second part) guarantees that $y^{(n)}$ also converges to zero. Leveraging this result and Lemma 4.1 (third part), we can further conclude that $x^{(n)}$ converges to g_0/g_1.

(iii). For any $n \in \mathbb{N}$, let $z^{(n)} > z^*$ for an initial point $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \in \Omega_1 \setminus \Omega_2$. Then, from $z^{(n)}$ is decreasing and bounded from below,

$$\lim_{n\to\infty} z^{(n)} \geq z^*.$$

Analysis of the third equation within the system (4.3) reveals that the convergence of the sequence $z^{(n)}$ necessitates the convergence of the sequence $x^{(n)}$. Analogously, examining the first equation within system (4.3) demonstrates that the convergence of $x^{(n)}$ necessitates the convergence of $y^{(n)}$. Consequently, based on the relationships defined in (4.3), we arrive at the following conclusions:

$$\lim_{n\to\infty} x^{(n)} = x^*, \ \lim_{n\to\infty} y^{(n)} = y^*, \ \lim_{n\to\infty} z^{(n)} = z^*.$$

□

For example, see Figure 2. Consider points $(0.5, 0.3, 0.016)$, and $(0.5, 0.18, 0.016)$ within set $\Omega_1 \setminus \Omega_2$. The trajectory of $(0.5, 0.18, 0.016)$ remains within $\Omega_1 \setminus \Omega_2$ and converges to $(1/2, 1/12, 3/200)$. However, the trajectory of $(0.5, 0.3, 0.016)$ leaves $\Omega_1 \setminus \Omega_2$ after a few steps and approaches $(2/3, 0, 0)$.

4.1. Biological interpretation

The model described in equation (1.2) characterizes the dynamic interactions of glucose, insulin, and β-cell mass. This section explores the predicted normal physiological behavior of the glucose regulatory system and investigates the potential pathways leading to diabetes, considering variations in model parameters and initial conditions.

To bridge the gap between the mathematical results and diabetes pathophysiology, we now provide a more detailed interpretation of the fixed points u_1^* and u_2^*, as well as the results of Theorems 4.1 and 4.2, in biological terms.

The fixed point u_2^* corresponds to a physiological (healthy) state in which glucose, insulin, and β-cell mass maintain stable, normal levels. This state reflects the homeostatic equilibrium typically observed in healthy individuals, where glucose levels are tightly regulated through feedback mechanisms involving insulin secretion by pancreatic β-cells. In this regime, small perturbations in glucose or insulin levels are corrected over time, indicating stability of the metabolic system.

On the other hand, the fixed point u_1^* corresponds to a pathological state characterized by high glucose concentration, near-zero insulin levels, and severely diminished or vanishing β-cell mass. This state resembles the clinical presentation of advanced diabetes mellitus, especially type 2 diabetes in its decompensated stage or type 1 diabetes, where pancreatic β-cells are destroyed or non-functional. As a result, the body fails to regulate glucose levels, leading to persistent hyperglycemia.

The parameters in our model, such as the death rate at zero glucose d_0 and the ratio of parameters r_1/r_2, can be interpreted as representing biological processes or external influences relevant to diabetes management and progression. For instance, an increase in d_0 may model enhanced β-cell apoptosis due to glucotoxicity or autoimmune attack, both of which are known risk factors for the development and worsening of diabetes. Conversely, therapeutic interventions aiming to reduce β-cell death or improve insulin sensitivity could be reflected as decreases in d_0 or modifications in r_1 and r_2.

Our analysis shows that when parameters satisfy $r_1^2 < 4r_2d_0$, the system tends toward the pathological fixed point u_1^*, characterized by elevated glucose levels, diminished insulin concentration, and reduced β-cell mass. Clinically, this state corresponds to diabetic hyperglycemia and progressive β-cell failure, hallmark features of type 2 diabetes. This implies that worsening parameters related to cell death and insulin production shifts the system into a disease state.

On the other hand, parameter regimes allowing the system to converge to the physiological fixed point u_2^* represent healthy glucose regulation with balanced insulin secretion and maintained β-cell mass. Understanding these parameter thresholds may help in designing targeted therapies to shift patients from pathological toward physiological states.

Interpretation of Theorem 4.1.

- Suppose that the parameter condition (2.1) holds and $r_1^2 < 4r_2d_0$. In this case, the death rate of β-cells at zero glucose (d_0) dominates the system dynamics. As a result, both the insulin level and the β-cell mass decrease over time, while the glucose level increases.
- Regardless of the initial condition $(x, y, z) \in \Omega$, the system eventually converges to the fixed point u_1^*, which reflects a pathological state characterized by β-cell failure and diabetic hyperglycemia.
- Biologically, this scenario corresponds to a situation where the compensatory mechanisms of the endocrine pancreas are insufficient to maintain glycemic control, leading to disease progression.

Interpretation of Theorem 4.2.

- Let (2.1) hold and suppose $r_1^2 = 4r_2d_0$. This is a borderline case between the regimes of physiological and pathological stability. Even if the glucose level remains above the threshold g_0/g_1, the system still evolves toward the pathological state u_1^*.
- If the initial β-cell mass is less than a critical threshold z^*, then the trajectory of the system moves toward the pathological fixed point. This suggests that inadequate initial β-cell mass may be a decisive factor leading to diabetes, even if glucose availability is sufficient.
- On the other hand, if theβ-cell mass remains strictly above z^* throughout the system's evolution, the system approaches the physiological fixed point u_2^*.

This indicates that sufficient β-cell reserve can stabilize insulin production and glycemic control.

In summary, the results show that both parameter values and initial conditions play a critical role in determining whether the system evolves toward health or disease. Theorems 4.1 and 4.2 mathematically validate that β-cell viability is a crucial determinant of diabetes progression, providing insight into the mechanisms of metabolic stability and failure.

Additionally, the model exhibits transient dynamics characterized by oscillations in glucose, insulin, and β-cell mass levels before eventual convergence to a steady state. These oscillations reflect the biological reality of glucose homeostasis, where the system responds dynamically to perturbations such as meal intake or insulin release.

Transient oscillations can be interpreted as the body's attempt to restore metabolic balance through feedback regulation mechanisms. In healthy individuals, such fluctuations are typically damped and return to equilibrium efficiently, preserving glucose stability.

However, in pathological parameter regimes or with insufficient βa-cell mass, these oscillations may become prolonged or amplified, indicating instability in glucose regulation. This can manifest clinically as erratic blood glucose levels, contributing to the onset or progression of diabetes.

Therefore, analyzing transient dynamics enriches the understanding of glucose-insulin-beta cell interactions, highlighting the importance of both steady-state behavior and temporal responses in metabolic health and disease.

Finally, our mathematical findings offer a framework to quantitatively link changes in biological parameters to clinical outcomes, thereby providing potential insights into the mechanisms of diabetes progression and strategies for intervention.

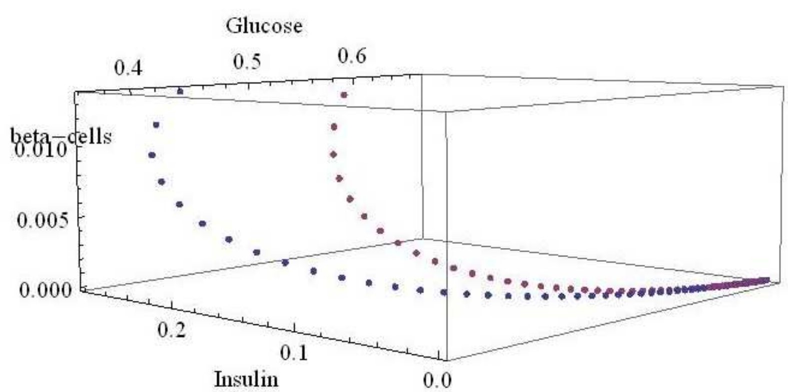

Figure 1. Given the system described by equation (1.2) and the following parameter values: $g_0 = 0.1$, $g_1 = 0.15$, $c = 0.6$, $s_1 = 1$, $s_2 = 0.2$, $k = 0.1$, $r_1 = r_2 = 1$, and $d_0 = 0.4$, where the inequality $r_1^2 < 4r_2d_0$ is satisfied, the system's trajectory converges to the pathological fixed point $(2/3, 0, 0)$ for both initial conditions $(0.5, 0.3, 0.016)$ and $(0.5, 0.18, 0.016)$. Here, glucose concentration is expressed in normalized units (1 unit $\approx$ 180 mg/dL), insulin in relative activity units, and β-cell mass as a fraction of its physiological maximum. These results illustrate that under high β-cell death rates at zero glucose, the system loses its capacity to maintain glucose homeostasis, regardless of moderate differences in initial insulin levels.

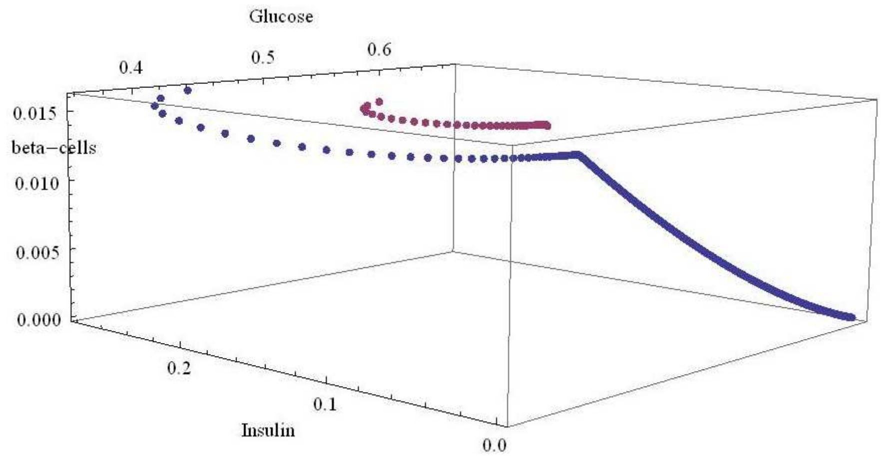

Figure 2. Given the system described by equation (1.2) with the parameters: $g_0 = 0.1$, $g_1 = 0.15$, $c = 0.6$, $s_1 = 1$, $s_2 = 0.2$, $k = 0.1$, $r_1 = r_2 = 1$, and $d_0 = 0.25$ (where the critical condition $r_1^2 = 4r_2d_0$ holds), we observed the following dynamical behavior. From the initial point $(x_0, y_0, z_0) = (0.5, 0.3, 0.016)$ — corresponding to a normoglycemic glucose level of approximately 90 mg/dL, moderate insulin concentration, and low β-cell mass — the system's trajectory converged to the pathological fixed point $(2/3, 0, 0)$. In contrast, starting from $(0.5, 0.18, 0.016)$, the system evolved toward the physiological fixed point $(1/2, 1/12, 3/200)$, indicating stable glucose regulation. Glucose is expressed in normalized units ($1 \approx 180$ mg/dL), insulin in arbitrary units, and β-cell mass as a relative proportion of its physiological maximum. These simulations reflect how subtle changes in initial conditions, particularly insulin levels, can influence long-term metabolic stability.

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Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and materials

The data sets generated during and/or analyzed during the current study are available from the author on reasonable request.

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