Journal of Nonlinear Modeling and Analysis https://global-sci.com/jnma
Volume 8, Number 2, April 2026, 494–519 DOI:10.12150/jnma.2026.494

# Numerical Solutions of Fuzzy Fractional Variable-Order Differential Equations

Ghadah S. E. Noman[1,2,†] and D. D. Pawar[2,†]

Received 21 February 2025; Accepted 7 July 2025

**Abstract** This paper addresses a class of fuzzy fractional differential equations (FFDEs) with variable-order (VO) derivatives, where the variable-order derivative is defined in the Caputo sense for fuzzy-valued functions. Using the $\Upsilon-$cut representation of fuzzy-valued functions, the original problem is reformulated into a new problem. To solve it, we apply operational matrices (OMs) derived from shifted Chebyshev polynomials of the third kind (SCP3). By approximating the unknown function and its derivative with SCP3, the problem is reduced to a system of nonlinear algebraic equations. A theoretical error analysis of the numerical solution is presented, along with an example to validate the method's accuracy.

**Keywords** Fuzzy fractional differential equations, variable-order, shifted Chebyshev polynomials of the third kind, operational matrix

**MSC(2010)** 26A33, 42C05, 34A07.

## 1. Introduction

Fractional calculus (FC) has emerged as a powerful mathematical framework for modeling complex and memory-dependent phenomena in diverse fields such as damping, viscoelasticity, wave propagation, diffusion, control systems, and signal processing [1–3]. The widespread applicability of fractional differential equations (FDEs) has led to the development of a broad range of analytical and numerical methods to solve them efficiently [4–6]. These methods not only enhance our understanding of such systems but also pave the way for developing more advanced computational techniques.

To further improve modeling accuracy, FDEs have been generalized through the introduction of VO operators, in which the order of differentiation or integration varies as a function of the independent variable. Several VO formulations have been proposed, including the Riemann-Liouville (RL) [7], Coimbra [9], Caputo–Fabrizio (CF) [8], and Atangana-Baleanu (AB) [10] operators. These provide greater flexibility for modeling processes with evolving memory and hereditary behavior.

[†]the corresponding author.
Email address:ghadah.sultan.esmail@gmail.com(G. S. E. Noman), dypawar@yahoo.com(D. D. Pawar)
[1]Department of Mathematics, Taiz University, Taiz-967, Yemen
[2]School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded-431606,India.

Despite their potential, VO-FDEs pose significant computational challenges due to the complexity of their integral kernels. In response, various numerical strategies have been proposed. Ganji et al. [10, 11] developed orthogonal polynomial-based techniques, while Bernstein polynomials have been employed in solving variable-order diffusion-wave equations [12]. More recently, Tingting et al. [13] used operational matrices with collocation to solve VO partial differential equations (PDEs), Jafari et al. [14] applied Hosoya polynomials to stochastic VO problems, Saha et al. [15] extended these approaches to VO-FDEs, and Mohd et al. [16] proposed a Bernstein-based method for nonlinear coupled reaction-diffusion systems.

In parallel, uncertainty modeling has progressed through the development of FFDEs, which integrate fuzzy logic into fractional calculus. FFDEs account for imprecision in system parameters, initial conditions, or boundary values using fuzzy numbers characterized by normality, convexity, upper semi-continuity, and compact support [17]. Numerous techniques have been proposed to solve FFDEs, including fuzzy Laplace transforms [18], Mittag-Leffler functions [19], fractional Euler methods [20], spline collocation [21], Chebyshev-based approaches [22], differential transforms [23], spectral methods [24]. Notable examples also include power series methods for fuzzy logistic equations [25] and fuzzy modeling in drug administration systems [26].

Combining the flexibility of VO operators with the uncertainty-handling capability of fuzzy sets leads to fuzzy variable-order fractional differential equations (VO-FFDEs). These hybrid models are well suited to capture both time-dependent memory effects and data uncertainty, making them applicable in engineering, physics, biology, finance, and other fields. Despite their promise, VO-FFDEs remain relatively underexplored, and challenges persist in their numerical formulation, stability, and implementation.

To contribute to this evolving area, Jafari et al. [27] proposed an approach using operational matrices based on shifted Legendre polynomials to solve fuzzy VO-FDEs involving Mittag-Leffler kernels.

In this study, we propose a spectral collocation method for solving VO-FFDEs using operational matrices based on shifted Chebyshev polynomials of the third kind. SCP3 polynomials are chosen for their orthogonality, computational efficiency, and strong endpoint clustering on $[0, 1]$, which improves accuracy near boundaries particularly important for VO-FDEs with non-uniform kernel behavior. Their structure also enables well-conditioned and sparse operational matrices, enhancing numerical stability.

The proposed method reformulates the fuzzy VO-FDE into a system of algebraic equations using the $\Upsilon$-cut representation and SCP3 based operational matrices. The resulting scheme is general, computationally efficient, and capable of handling linear fuzzy systems.

We consider the following form of a fuzzy variable-order fractional differential equation:

$$\begin{cases} {}_0^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)} \mathsf{U}(\mathtt{t}_1) = \mathfrak{f}(\mathtt{t}_1, \mathsf{U}(\mathtt{t}_1)), \\ \\ \qquad \mathsf{U}(0) = \mathsf{U}_0 \in \mathbb{E}_{\mathrm{F}}. \end{cases} \tag{1.1}$$

Here, $0 < \varsigma(\mathtt{t}_1) < 1$, and $\mathbb{E}_{\mathrm{F}}$ represents the space of fuzzy numbers. The function $\mathfrak{f} : [0, 1] \times \mathbb{E}_{\mathrm{F}} \to \mathbb{E}_{\mathrm{F}}$ is a continuous fuzzy function, and the unknown function $\mathsf{U}(\mathtt{t}_1)$ belongs to $C_{\mathbb{E}_{\mathrm{F}}}[0,1] \cap L_{\mathbb{E}_{\mathrm{F}}}[0,1]$ . The term ${}^{C}_{0}D^{\varsigma(\mathtt{t}_1)}_{\mathtt{t}_1}$ denotes the fuzzy Caputo derivative.

The paper is structured as follows: Section 2 introduces the fundamental definitions in the calculus of fuzzy sets and functions. Section 3 outlines the definitions and properties of SCP3, as well as the approximation of fuzzy functions and computation of operational matrices of derivatives using SCP3. Section 4 introduces a numerical method that reformulates problem in Eq.(1.1) with an initial condition in terms of fuzzy numbers, converting it into a system of nonlinear algebraic equations. Section 5 provides error analysis and includes illustrative examples. Lastly, Section 6 summarizes the findings and offers concluding remarks.

# 2. Preliminaries and notations

Here, we present key foundational definitions relevant to fuzzy theory.

## 2.1. Fuzzy calculus

**Definition 2.1.** [28] A fuzzy set $\aleph$ within $\mathbb{E}_{\mathrm{F}}$ is defined as a function

$$\aleph : \mathbb{E}_{\mathrm{F}} \to [0,1],$$

where $\aleph(\mathtt{t}_1)$ represents the membership degree of $\mathtt{t}_1$ in $\aleph$.

The support of $\aleph$ is defined as:

$$\mathrm{supp}(\aleph) = \{\mathtt{t}_1 \in \mathbb{E}_{\mathrm{F}} \mid \aleph(\mathtt{t}_1) > 0\},$$

where $\mathbb{E}_{\mathrm{F}}$ denotes the set of all fuzzy numbers on $\mathbb{R}$.

**Definition 2.2.** [29] A fuzzy number $\aleph : \mathbb{R} \to [0,1]$ satisfies:

**(1)** Normality: For $\mathtt{t}_0 \in \mathbb{R}, \aleph$ is normal, meaning $\aleph(\mathtt{t}_0) = 1$;

**(2)** Convexity: For $\mathtt{t}_1, \mathtt{t}_2, \in \mathbb{R}$ and $r \in [0,1]$, $\aleph$ is convex, satisfying

$$\aleph(r\mathtt{t}_1 + (1-r)\mathtt{t}_2) \geq \min\{\aleph(\mathtt{t}_1), \aleph(\mathtt{t}_2)\};$$

**(3)** Upper semi-continuity: $\aleph$ is upper semi-continuous;

**(4)** Compact Support: $cl\{\mathtt{t}_1 \in \mathbb{R}, \aleph(\mathtt{t}_1) > 0\}$ is compact.

The definition of a fuzzy number is not unique and varies across contexts. While the classical form includes normality, convexity, upper semi-continuity, and compact support, alternative definitions address pathological cases and discontinuities. Notable generalizations are discussed in [30,31], offering broader applicability in fuzzy fractional differential equations.

**Definition 2.3.** A fuzzy number $\aleph$ can be represented in parametric interval form as $[\aleph]^{\curlyvee} = [\underline{\aleph}(\curlyvee), \overline{\aleph}(\curlyvee)]$, for $0 \leq \curlyvee \leq 1$, if and only if

**(i)** $\underline{\aleph}(\curlyvee)$is an increasing, bounded, and left-continuous function on (0,1];

**(ii)** $\overline{\aleph}(\curlyvee)$ is a decreasing, bounded, and right-continuous function on (0,1];

**(iii)** $\underline{\aleph}(\curlyvee) \leq \overline{\aleph}(\curlyvee)$ holds for all.

For a fuzzy-valued function $\aleph : A \subseteq \mathbb{R} \to \mathbb{E}_{\mathrm{F}}$. Its $\curlyvee$-cut are closed intervals in $\mathbb{R}$, represented as:

$$[\aleph(\mathtt{t}_1, \curlyvee)] = [\underline{\aleph}(\mathtt{t}_1, \curlyvee), \overline{\aleph}(\mathtt{t}_1, \curlyvee)].$$

**Definition 2.4.** [32, 33] Let $\mathfrak{k} \in \mathbb{R}$, $[\mathsf{U}]^{\curlyvee} = [\underline{\mathsf{U}}, \overline{\mathsf{U}}]$ and $[\mathsf{z}]^{\curlyvee} = [\underline{\mathsf{z}}, \overline{\mathsf{z}}]$. The following properties hold:

(1) $(\mathsf{U} \oplus \mathsf{z}) = [\underline{\mathsf{U}} + \underline{\mathsf{z}}, \overline{\mathsf{U}} + \overline{\mathsf{z}}]$

(2) $\mathfrak{k} \odot \mathsf{U} = \begin{cases} [\mathfrak{k}\underline{\mathsf{U}}, \overline{\mathsf{U}}], \mathfrak{k} \geq 0, \\ [\mathfrak{k}\overline{\mathsf{U}}, \mathfrak{k}\underline{\mathsf{U}}], \mathfrak{k} < 0 \end{cases}$

(3) The Hausdorff distance is defined as follows:

$$\mathbf{D} : \mathbb{E}_{\mathrm{F}} \times \mathbb{E}_{\mathrm{F}} \to \mathbb{R}^{+} \cup \{0\},$$
$$\mathbf{D}(\mathsf{U}, \mathsf{z}) = \sup_{\curlyvee \in [0,1]} \max\{|\underline{\mathsf{U}} - \underline{\mathsf{z}}|, |\overline{\mathsf{U}} - \overline{\mathsf{z}}|\},$$

where this distance evaluates the supremum of the pointwise distances between functions $\mathsf{U}$ and $\mathsf{z}$ across the interval [0,1], we establish the relevant properties of this metric as detailed in [32].

(4) The uniform distance is represented by:

$$\mathbf{D}^{*}(\mathsf{U}, \mathsf{z}) = \sup_{\mathtt{t}_1 \in [0,1]} \mathbf{D}(\mathsf{U}(\mathtt{t}_1), \mathsf{z}(\mathtt{t}_1)).$$

**Definition 2.5.** Let $\mathsf{u}, \mathsf{v} \in \mathbb{E}_{\mathrm{F}}$, if there exists $\mathsf{w} \in \mathbb{E}_{\mathrm{F}}$ such that $\mathsf{u} = \mathsf{v} \oplus \mathsf{w}$ then $\mathsf{w}$ is called the Hukuhara difference of $\mathsf{u}$ and $\mathsf{v}$ and it is denoted by $\mathsf{u} \ominus \mathsf{v}$.

**Definition 2.6.** [34] Let $\mathsf{u}$ and $\mathsf{v}$ be two fuzzy numbers. Then, the generalized Hukuhara difference of these two fuzzy numbers is given as follows

$$\mathsf{u} \ominus_{gH} \mathsf{v} = \mathsf{w} \Leftrightarrow (i)\mathsf{u} = \mathsf{v} \oplus \mathsf{w} \text{ or } (ii)\mathsf{v} = \mathsf{u} \oplus (-1)\mathsf{w}.$$

**Definition 2.7.** [32] Let $\tilde{a} \in (0, 1)$ and let $\hbar$ be such that $\tilde{a} + \hbar \in (0, 1)$. The generalized Hukuhara derivative of a fuzzy-valued function $\aleph : (0, 1) \to \mathbb{E}_{\mathrm{F}}$ at $\tilde{a}$ is defined by:

$$\aleph'(\tilde{a}) = \lim_{\hbar \to 0^{+}} \frac{\aleph\left(\tilde{a} + \hbar\right) \ominus \aleph\left(\tilde{a}\right)}{\hbar}.$$

If $\aleph'(\tilde{a})$ exists for all $\tilde{a} \in (0, 1)$, then $\aleph$ is said to be generalized Hukuhara differentiable (gH-differentiable) on (0, 1).

**Definition 2.8.** [32] Let $\aleph : (0, 1) \to \mathbb{E}_{\mathrm{F}}$ and suppose $\tilde{a} \in (0, 1)$. Then, $\aleph$ is differentiable at $\tilde{a}$, if one of the following conditions is satisfied:

(1) There exists a derivative $\aleph'(\tilde{a}) \in \mathbb{E}_{\mathrm{F}}$, such that, for all $\hbar > 0$, the expressions $\aleph(\tilde{a} + \hbar) \ominus \aleph(\tilde{a})$, and $\aleph(\tilde{a}) \ominus \aleph(\tilde{a} - \hbar)$ exist, and the following limits (in a fuzzy metric) hold:

$$\aleph'(\tilde{a}) = \lim_{\hbar \to 0^{+}} \frac{\aleph(\tilde{a} + \hbar) \ominus \aleph(\tilde{a})}{\hbar}$$
$$= \lim_{\hbar \to 0^{+}} \frac{\aleph(\tilde{a}) \ominus \aleph(\tilde{a} - \hbar)}{\hbar}.$$

(2) There exists a derivative $\aleph'(\tilde{a}) \in \mathbb{E}_{\mathrm{F}}$, such that, for all $\hbar < 0$, the expressions $\aleph(\tilde{a}+\hbar) \ominus \aleph(\tilde{a})$, and $\aleph(\tilde{a}) \ominus \aleph(\tilde{a}-\hbar)$ exist, and the following limits (in a fuzzy metric) hold:

$$\begin{aligned}\aleph'(\tilde{a}) &= \lim_{\hbar \to 0^-} \frac{\aleph(\tilde{a}+\hbar) \ominus \aleph(\tilde{a})}{\hbar} \\ &= \lim_{\hbar \to 0^-} \frac{\aleph(\tilde{a}) \ominus \aleph(\tilde{a}-\hbar)}{\hbar}.\end{aligned}$$

Now, if $\aleph = [\underline{\aleph}, \overline{\aleph}]$ is differentiable in the first form (1) ((i)-differentiable), then $\aleph' = [\underline{\aleph}', \overline{\aleph}']$. Similarly, if $\aleph$ is differentiable in the second form (2) ((ii)-differentiable), then $\aleph' = [\overline{\aleph}', \underline{\aleph}']$.

## 2.2. Fractional fuzzy differentiation

**Definition 2.9.** Let $\aleph \in C_{\mathbb{E}_{\mathrm{F}}}[0,1] \cap L_{\mathbb{E}_{\mathrm{F}}}[0,1]$, and $\mathtt{t}_1, \curlyvee \in [0,1]$. The fuzzy RL-integral of order $\varsigma(\mathtt{t}_1)$, $0 < \varsigma(\mathtt{t}_1) < 1$ is defined as

(i) If $\aleph(\mathtt{t}_1, \curlyvee)$ is (i)-differentiable, then

$$^{RL}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1, \curlyvee) = \left[{}^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1, \curlyvee), {}^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1, \curlyvee)\right].$$

(ii) If $\aleph(\mathtt{t}_1, \curlyvee)$ is (ii)-differentiable, then

$$^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1, \curlyvee) = \left[{}^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1, \curlyvee), {}^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1, \curlyvee)\right],$$

where

$$\begin{aligned}{}^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1, \curlyvee) &= \frac{1}{\Gamma(\varsigma(\mathtt{t}_1))}\int_0^{\mathtt{t}_1} (\mathtt{t}_1 - \mathrm{s})^{\varsigma(\mathtt{t}_1)-1}\underline{\aleph}(\mathrm{s})d\mathrm{s}. \\ {}^{\mathrm{RL}}I_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1, \curlyvee) &= \frac{1}{\Gamma(\varsigma(\mathtt{t}_1))}\int_0^{\mathtt{t}_1} (\mathtt{t}_1 - \mathrm{s})^{\varsigma(\mathtt{t}_1)-1}\overline{\aleph}(\mathrm{s})d\mathrm{s}.\end{aligned}$$

**Definition 2.10.** Let $\aleph \in C_{\mathbb{E}_{\mathrm{F}}}[0,1] \cap L_{\mathbb{E}_{\mathrm{F}}}[0,1]$, and $\mathtt{t}_1, \curlyvee \in [0,1]$. The fuzzy Caputo derivative of order $\varsigma(\mathtt{t}_1)$, $0 < \varsigma(\mathtt{t}_1) < 1$ is defined as

(i) If $\aleph(\mathtt{t}_1, \curlyvee)$ is (i)-differentiable, then

$$^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1, \curlyvee) = \left[{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1, \curlyvee), {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1, \curlyvee)\right].$$

(ii) If $\aleph(\mathtt{t}_1, \curlyvee)$ is (ii)-differentiable, then

$$^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1, \curlyvee) = \left[{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1, \curlyvee), {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1, \curlyvee)\right],$$

where

$$\begin{aligned}{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1, \curlyvee) &= \frac{1}{\Gamma(1-\varsigma(\mathtt{t}_1))}\int_0^{\mathtt{t}_1} (\mathtt{t}_1 - \mathrm{s})^{-\varsigma(\mathtt{t}_1)}\underline{\aleph}'(\mathrm{s}, \curlyvee)d\mathrm{s}. \\ {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1, \curlyvee) &= \frac{1}{\Gamma(1-\varsigma(\mathtt{t}_1))}\int_0^{\mathtt{t}_1} (\mathtt{t}_1 - \mathrm{s})^{-\varsigma(\mathtt{t}_1)}\overline{\aleph}'(\mathrm{s}, \curlyvee)d\mathrm{s},\end{aligned}$$

which satisfies the following property [35]:

$$
{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathtt{t_1}^{r} = \left\{ \begin{array}{cc} 0 & \text{for } r = 0, \\ \frac{\Gamma(r+1)}{\Gamma(r+1-\varsigma(\mathtt{t}_1))}\mathtt{t_1}^{r-\varsigma(\mathtt{t}_1)} & \text{for } r = 1, 2, \dots \end{array} \right\} \tag{2.1}
$$

**Theorem 2.1.** *Let $\aleph$, and $\mathscr{X}$ represent two fuzzy-valued functions. The following inequality holds for values in the interval [0, 1].*

$$
\left| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t_1}) \right| \leqslant \Theta_{\Upsilon} \left\| \aleph(\mathtt{t_1}) - \mathscr{X}(\mathtt{t_1}) \right\|,
$$

*where* $\Theta_{\Upsilon} = \frac{T^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(2-\varsigma(\mathtt{t}_1))}$.

**Proof.** Based on Definition 2.9, we can express the following relationship:

$$
\begin{aligned}
&\left| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t_1}) \right| \\
&= \left[ \left| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathscr{X}}(\mathtt{t_1}) \right|, \left| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathscr{X}}(\mathtt{t_1}) \right| \right].
\end{aligned}
$$

Next, for the term involving the lower bound of the fuzzy functions, we have:

$$
\begin{aligned}
&\left| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathscr{X}}(\mathtt{t_1}) \right| \\
&= \left| \frac{1}{\Gamma(1-\varsigma(\mathtt{t_1}))} \int_0^{\mathtt{t}_1} (\mathtt{t_1} - \mathrm{s})^{-\varsigma(\mathtt{t}_1)} \underline{\aleph}'(\mathrm{s}) ds - \frac{1}{\Gamma(1-\varsigma(\mathtt{t_1}))} \int_0^{\mathtt{t}_1} (\mathtt{t_1} - \mathrm{s})^{-\varsigma(\mathtt{t}_1)} \underline{\mathscr{X}}'(\mathrm{s}) ds \right| \\
&= \left| \frac{1}{\Gamma(1-\varsigma(\mathtt{t_1}))} \left( \int_0^{\mathtt{t}_1} (\mathtt{t_1} - \mathrm{s})^{-\varsigma(\mathtt{t}_1)} \underline{\aleph}'(\mathrm{s}) ds - \int_0^{\mathtt{t}_1} (\mathtt{t_1} - \mathrm{s})^{-\varsigma(\mathtt{t}_1)} \underline{\mathscr{X}}'(\mathrm{s}) ds \right) \right| \\
&\leqslant \frac{\mathfrak{T}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(1-\varsigma(\mathtt{t_1}))(1-\varsigma(\mathtt{t_1}))} \left| \int_0^{\mathtt{t}_1} \underline{\aleph}'(\mathrm{s}) ds - \int_0^{\mathtt{t}_1} \underline{\mathscr{X}}'(\mathrm{s}) ds \right| \\
&\leqslant \frac{\mathfrak{T}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(2-\varsigma(\mathtt{t_1})} \left\| \underline{\aleph}(\mathtt{t_1}) - \underline{\mathscr{X}}(\mathtt{t_1}) \right\| \leqslant \underline{\Theta}_{\Upsilon} \left\| \underline{\aleph}(\mathtt{t_1}) - \underline{\mathscr{X}}(\mathtt{t_1}) \right\|.
\end{aligned}
$$

This leads to the conclusion that

$$
\left\| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathscr{X}}(\mathtt{t_1}) \right\| \leqslant \underline{\Theta}_{\Upsilon} \left\| \underline{\aleph}(\mathtt{t_1}) - \underline{\mathscr{X}}(\mathtt{t_1}) \right\|.
$$

Similarly, for the upper part, we have

$$
\left\| {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t_1}) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathscr{X}}(\mathtt{t_1}) \right\| \leqslant \overline{\Theta}_{\Upsilon} \left\| \overline{\aleph}(\mathtt{t_1}) - \overline{\mathscr{X}}(\mathtt{t_1}) \right\|.
$$

□

**Theorem 2.2.** *Let $\aleph(t)$, $\mathscr{X}$ be two fuzzy-valued functions. The following inequalities can be derived:*

$$
\mathbf{D}^*({}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(t), {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t_1})) \leqslant \Theta_{\Upsilon} \left\| \aleph(\mathtt{t_1}) - \mathscr{X}(\mathtt{t_1}) \right\|,
$$

*where* $\Theta_{\Upsilon} = \max_{\Upsilon \in [0,1]} \{ \underline{\Theta}_{\Upsilon}, \overline{\Theta}_{\Upsilon} \}$.

**Proof.** According to Definition 2.4, we can express the following relationship:

$$\mathbf{D}^*({}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1), {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t}_1)) = \sup_{\mathtt{t}_1\in[0,1]} \mathbf{D}({}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1), {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t}_1)) =$$

$$\sup_{\mathtt{t}_1\in[0,1]}\ \sup_{\Upsilon\in[0,1]} \max\{|{}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1) -{}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathscr{X}}(\mathtt{t}_1)|, |{}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1) -{}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathscr{X}}(\mathtt{t}_1)|\}.$$

By applying Theorem 2.1, we obtain:

$$\begin{aligned}&\mathbf{D}^*({}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(t), {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t}_1))\\ &\leqslant \sup_{\Upsilon\in[0,1]} \max\{\underline{\Theta}_\Upsilon \left\|\underline{\aleph}(\mathtt{t}_1) - \underline{\mathscr{X}}(\mathtt{t}_1)\right\|, \overline{\Theta}_\Upsilon \left\|\overline{\aleph}(\mathtt{t}_1) - \overline{\mathscr{X}}(\mathtt{t}_1)\right\|\},\end{aligned}$$

where, $\Theta_\Upsilon = \max_{\Upsilon\in[0,1]}\{\underline{\Theta}_\Upsilon, \overline{\Theta}_\Upsilon\}$, and $||.||$ denotes the $\infty$ norm. □

# 3. Main result

## 3.1. Chebyshev polynomials and SCP3

The Chebyshev polynomials of the third kind, denoted by $\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1)$, form a set of orthogonal polynomials in the variable $\mathtt{t}_1$ with degree $\mathtt{m}$, and are defined over the interval [-1,1]. The expression for $\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1)$ is given by:

$$\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1) = \frac{\cos\left(\mathtt{m} + \frac{1}{2}\right)\theta}{\cos\left(\frac{\theta}{2}\right)},$$

where $\mathtt{t}_1 = \cos\theta$ and $\theta \in [0, \pi]$. Additionally, the polynomials $\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1)$ exhibit orthogonality on the interval [-1,1] with respect to the defined inner product.

$$< \mathscr{V}_{\mathtt{m}}(\mathtt{t}_1), \mathscr{V}_{\mathtt{n}}(\mathtt{t}_1) >= \int_{-1}^{1} \sqrt{\frac{1+\mathtt{t}_1}{1-\mathtt{t}_1}}\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1)\mathscr{V}_{\mathtt{n}}(\mathtt{t}_1)d\mathtt{t}_1 = \begin{cases}\pi\ \mathtt{m}=\mathtt{n},\\ 0\ \mathtt{m}\neq\mathtt{n}.\end{cases}$$

The weight function associated with the Chebyshev polynomials $\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1)$ of the third kind is given by $\sqrt{\frac{1+\mathtt{t}_1}{1-\mathtt{t}_1}}$. These polynomials can be constructed using a recurrence relation, starting from initial polynomial values, as described below:

$$\begin{aligned}&\mathscr{V}_0(\mathtt{t}_1) = 1,\\ &\mathscr{V}_1(\mathtt{t}_1) = 2\mathtt{t}_1 - 1,\\ &\mathscr{V}_{\mathtt{m}+1}(\mathtt{t}_1) = 2t\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1) - \mathscr{V}_{\mathtt{m}-1}(\mathtt{t}_1), \quad \mathtt{m} = 1, 2, \ldots\end{aligned}$$

The SCP3, denoted by $\mathscr{V}_{\mathtt{m}}(\mathtt{t}_1)$, are defined over the interval [0,1] with degree $\mathtt{m}$. For any arbitrary interval [0,1], they are expressed as

$$\mathscr{V}^*_{\mathtt{m}}(\mathtt{t}_1) = \mathscr{V}_{\mathtt{m}}(2\mathtt{t}_1 - 1).$$

These polynomials exhibit orthogonality within the interval [0,1] as follows:

$$< \mathscr{V}^*_{\mathtt{m}}(\mathtt{t}_1), \mathscr{V}^*_{\mathtt{n}}(\mathtt{t}_1) >= \int_{0}^{1} \mathscr{W}(\mathtt{t}_1)\mathscr{V}^*_{\mathtt{m}}(\mathtt{t}_1)\mathscr{V}^*_{\mathtt{n}}(\mathtt{t}_1)d\mathtt{t}_1 = \begin{cases}\pi/2\ \mathtt{m}=\mathtt{n},\\ 0\quad \mathtt{m}\neq\mathtt{n}.\end{cases}$$

The weight function associated with $\mathscr{V}_{\mathtt{m}}^*(\mathtt{t_1})$ is given by $\mathscr{W}(\mathtt{t_1}) = \sqrt{\frac{\mathtt{t_1}}{1-\mathtt{t_1}}}$ and is normalized to ensure that $\mathscr{V}_{\mathtt{m}}^*(1) = 1$. These polynomials can be generated using a recurrence relation, beginning with the initial values:

$$\begin{aligned}
&\mathscr{V}_0^*(\mathtt{t_1}) = 1,\\
&\mathscr{V}_1^*(\mathtt{t_1}) = 4\mathtt{t_1} - 3,\\
&\mathscr{V}_{\mathtt{m}+1}^*(\mathtt{t_1}) = 2(2\mathtt{t_1} - 1)\mathscr{V}_{\mathtt{m}}^*(\mathtt{t_1}) - \mathscr{V}_{\mathtt{m}-1}^*(\mathtt{t_1}), \quad \mathtt{m} = 1, 2, \ldots
\end{aligned}$$

The SCP3, can be represented by this analytical form.

$$\mathscr{V}_{\mathtt{m}}^*(\mathtt{t_1}) = \sum_{k=0}^{\mathtt{m}} (-1)^k 2^{2(\mathtt{m}-k)} \frac{(2\mathtt{m}+1)\Gamma(2\mathtt{m}-k+1)}{(2k+1)\Gamma(2\mathtt{m}-2k+2)} \mathtt{t_1}^{(\mathtt{m}-k)}, \quad \mathtt{m} \in \mathbb{Z}^+. \tag{3.1}$$

## 3.2. Function approximation

Research indicates that the set of SCP3 forms a complete basis in the Hilbert space $L^2[0,1]$. Consequently, any function $\mathsf{U}(\mathtt{t_1})$ in $L^2[0,1]$ can be approximated using SCP3, as shown in [36].

$$\mathsf{U}(\mathtt{t_1}) = \sum_{r=0}^{\infty} c_r \mathscr{V}_r^*(\mathtt{t_1}).$$

The polynomial coefficients are represented by $c_r$. If $\mathsf{U}(\mathtt{t_1})$ is a polynomial of degree $\mathtt{m}$, it can be expressed using the SCP3 as follows:

$$\mathsf{U}(\mathtt{t_1}) \cong \mathsf{U}_{\mathtt{m}}(\mathtt{t_1}) = \sum_{r=0}^{\mathtt{m}} c_r \mathscr{V}_r^*(\mathtt{t_1}) = \mathsf{C}^T \vartheta(\mathtt{t_1}). \tag{3.2}$$

The vector $\mathsf{C} = [c_0, c_1, \ldots, c_{\mathtt{m}}]^T$ represents the coefficient vector of length of $\mathtt{m}+1$. This vector can be determined using the following integration technique:

$$c_r = \frac{2}{\pi} \int_0^1 \mathsf{U}(\mathtt{t_1}) \mathscr{W}(\mathtt{t_1}) \mathscr{V}_r^*(\mathtt{t_1}) d\mathtt{t_1}.$$

The SCP3 vector $\vartheta(\mathtt{t_1})$ can be represented in matrix form as:

$$\vartheta(\mathtt{t_1}) = \mathsf{P}\mathtt{T}(\mathtt{t_1}), \tag{3.3}$$

where

$$\begin{aligned}
\vartheta(\mathtt{t_1}) &= [\mathscr{V}_0^*(\mathtt{t_1}), \mathscr{V}_1^*(\mathtt{t_1}), \ldots, \mathscr{V}_{\mathtt{m}}^*(\mathtt{t_1})]^T,\\
\mathtt{T}(\mathtt{t_1}) &= \left[1, \mathtt{t_1}, \mathtt{t_1}^2, \ldots, \mathtt{t_1}^{\mathtt{m}}\right]^T.
\end{aligned}$$

The matrix $\mathsf{P}$ is a lower-triangle square matrix of dimensions $(\mathtt{m}+1) \times (\mathtt{m}+1)$, and holds the coefficient values for the SCP3. For instance, when $\mathtt{m} = 4$, the square matrix $\mathsf{P}$ is represented as follows:

$$\mathsf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 \\ 9 & -120 & 432 & -576 & 256 \end{bmatrix}.$$

Hence, by employing Eq. (3.3), we can reformulate:

$$\mathtt{T}(\mathtt{t_1}) = \mathsf{P}^{-1}\vartheta(\mathtt{t_1}). \tag{3.4}$$

### 3.3. Fuzzy function approximation

Consider a function $\mathsf{U}(\mathtt{t}_1)$ that belongs to the space $C_{\mathbb{E}_\mathrm{F}}[0,1] \cap L_{\mathbb{E}_\mathrm{F}}[0,1]$, and let $\mathscr{V}_r^*(\mathtt{t}_1)$ be the SCP3, which are real-valued functions defined on the interval [0, 1]. The goal is to find the fuzzy approximation of the function $\mathsf{U}_\mathtt{m}(\mathtt{t}_1)$, expressed similarly to the definition as:

$$\mathsf{U}(\mathtt{t}_1) = \sum_{r=0}^{\infty} c_r \odot \mathscr{V}_r^*(\mathtt{t}_1),$$

where the fuzzy coefficients $c_r$ are given by:

$$c_r = \frac{2}{\pi} \odot \int_0^1 \mathsf{U}(\mathtt{t}_1) \odot \mathscr{W}(\mathtt{t}_1) \odot \mathscr{V}_r^*(\mathtt{t}_1) d\mathtt{t}_1.$$

To approximate the function $\mathsf{U}(\mathtt{t}_1)$, we use the first $(\mathtt{m}+1)$ terms of the series:

$$\mathsf{U}(\mathtt{t}_1) \cong \mathsf{U}_\mathtt{m}(\mathtt{t}_1) = \sum_{r=0}^{\mathtt{m}} c_r \odot \mathscr{V}_r^*(\mathtt{t}_1) = \mathsf{C}_\curlyvee^T \odot \vartheta(\mathtt{t}_1), \tag{3.5}$$

where $\mathsf{C}_\curlyvee = \left[\underline{\mathsf{C}}_\curlyvee, \overline{\mathsf{C}}_\curlyvee\right]$ is the fuzzy coefficient vector and $\vartheta(\mathtt{t}_1)$ is the vector of SCP3.

$$\vartheta(\mathtt{t}_1) = \left[\mathscr{V}_0^*(\mathtt{t}_1), \mathscr{V}_1^*(\mathtt{t}_1), \ldots, \mathscr{V}_\mathtt{m}^*(\mathtt{t}_1)\right]^T,$$

$$\mathsf{C}_\curlyvee = \left[c_0, c_1, \ldots, c_\mathtt{m}\right]^T.$$

Thus, the fuzzy approximation of $\mathsf{U}(\mathtt{t}_1)$ is given by:

$$\mathsf{U}(\mathtt{t}_1, \curlyvee) = [\underline{\mathsf{U}}(\mathtt{t}_1, \curlyvee), \overline{\mathsf{U}}(\mathtt{t}_1, \curlyvee)] = \left[\sum_{r=0}^{\mathtt{m}} \underline{c}_r(\curlyvee)\mathscr{V}_r^*(\mathtt{t}_1), \sum_{r=0}^{\mathtt{m}} \overline{c}_r(\curlyvee)\mathscr{V}_r^*(\mathtt{t}_1)\right].$$

This can be expressed as:

$$\mathsf{U}(\mathtt{t}_1, \curlyvee) = \left[\underline{\mathsf{C}}_\curlyvee^T \vartheta(\mathtt{t}_1), \overline{\mathsf{C}}_\curlyvee^T \vartheta(\mathtt{t}_1)\right]. \tag{3.6}$$

**Theorem 3.1.** *Assume that the function $\mathsf{U}(\mathtt{t}_1)$ is continuously differentiable up to the $\mathtt{m}$-th order on the interval [0,1]. The best square approximation of $\mathsf{U}(\mathtt{t}_1)$, denoted by $\mathsf{Um}(\mathtt{t}_1)$ defined in Eq(3.5), satisfies the following inequality:*

$$\left\|\mathsf{U}(\mathtt{t}_1) - \tilde{\mathsf{U}}_\mathtt{m}(\mathtt{t}_1)\right\| \leqslant \frac{M(\mathsf{K})^{\mathtt{m}+1}}{\Gamma(\mathtt{m})}\sqrt{\frac{\pi}{2}},$$

*$M$ represents the maximum value of $\mathsf{U}^{(\mathtt{m}+1)}(\mathtt{t}_1)$ over the interval $[0,1]$ and $\mathsf{K}$ is determined as* $\max\{1-\mathtt{t}_0, \mathtt{t}_0\}$.

**Proof.** Using the Taylor expansion, the function $\mathtt{U}(\mathtt{t}_1,\Upsilon) = [\underline{\mathtt{U}}(\mathtt{t}_1,\Upsilon),\overline{\mathtt{U}}(\mathtt{t}_1,\Upsilon)]$ can be expanded as:

$$\underline{\mathtt{U}}(\mathtt{t}_1,\Upsilon) = \sum_{\mathtt{q}=0}^{\mathtt{m}} \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{q}}}{\Gamma(\mathtt{q}-1)} \underline{\mathtt{U}}^{(q)}(\mathtt{t}_0) + \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{m}+1}}{\Gamma(\mathtt{m})} \underline{\mathtt{U}}^{(\mathtt{m}+1)}(\nu),$$

and

$$\underline{\mathtt{U}}(\mathtt{t}_1,\Upsilon) = \sum_{\mathtt{q}=0}^{\mathtt{m}} \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{q}}}{\Gamma(\mathtt{q}-1)} \overline{\mathtt{U}}^{(q)}(\mathtt{t}_0) + \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{m}+1}}{\Gamma(\mathtt{m})} \overline{\mathtt{U}}^{(\mathtt{m}+1)}(\nu).$$

Let $\mathtt{t}_0 \in [0,1]$ and $\nu$ is a value between $\mathtt{t}_0$ and $\mathtt{t}_1$. Now, we define the approximations:

$$\underline{\tilde{\mathtt{U}}}_{\mathtt{m}}(\mathtt{t}_1,\Upsilon) = \sum_{\mathtt{q}=0}^{\mathtt{m}} \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{q}}}{\Gamma(\mathtt{q}-1)} \underline{\mathtt{U}}^{(q)}(\mathtt{t}_0),$$

and

$$\overline{\tilde{\mathtt{U}}}_{\mathtt{m}}(\mathtt{t}_1,\Upsilon) = \sum_{\mathtt{q}=0}^{\mathtt{m}} \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{q}}}{\Gamma(\mathtt{q}-1)} \overline{\mathtt{U}}^{(q)}(\mathtt{t}_0).$$

Next, we estimate the error between the actual and the approximated functions:

$$\left\| \underline{\mathtt{U}}(\mathtt{t}_1) - \underline{\tilde{\mathtt{U}}}_{\mathtt{m}}(\mathtt{t}_1) \right\| = \left| \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{m}+1}}{\Gamma(\mathtt{m})} \underline{\mathtt{U}}^{(\mathtt{m}+1)}(\nu) \right|.$$

Since $\underline{\mathtt{U}}_{\mathtt{m}}(\mathtt{t}_1)$ represents the best square approximation of $\underline{\mathtt{U}}(\mathtt{t})$, we can deduce:

$$\begin{aligned}
&\|\underline{\mathtt{U}}(\mathtt{t}_1) - \underline{\mathtt{U}}_{\mathtt{m}}(\mathtt{t}_1)\|^2 \leqslant \left\| \underline{\mathtt{U}}(\mathtt{t}_1) - \underline{\tilde{\mathtt{U}}}_{\mathtt{m}}(\mathtt{t}_1) \right\|^2 \\
&= \int_0^1 \underline{\mathscr{W}}(\mathtt{t}_1) \left[ \underline{\mathtt{U}}(\mathtt{t}_1) - \underline{\tilde{\mathtt{U}}}_{\mathtt{m}}(\mathtt{t}_1) \right]^2 d\mathtt{t}_1 \\
&= \int_0^1 \underline{\mathscr{W}}(\mathtt{t}_1) \left[ \frac{(\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{m}+1}}{\Gamma(\mathtt{m})} \underline{\mathtt{U}}^{(\mathtt{m}+1)}(\nu) \right]^2 d\mathtt{t}_1 \\
&\leqslant \frac{\underline{M}^2}{[\Gamma(\mathtt{m})]^2} \int_0^1 \underline{\mathscr{W}}(\mathtt{t}_1) \left[ (\mathtt{t}_1-\mathtt{t}_0)^{\mathtt{m}+1} \right]^2 d\mathtt{t}_1,
\end{aligned}$$

where $\underline{M} = \max_{\nu\in[0,1]} |\underline{\mathtt{U}}^{(m+1)}(\nu)|$. Now, let $\underline{\mathtt{K}} = \max\{1-\mathtt{t}_0, \mathtt{t}_0\}$. Then

$$\begin{aligned}
&\|\underline{\mathtt{U}}(\mathtt{t}_1) - \underline{\mathtt{U}}_{\mathtt{m}}(\mathtt{t}_1)\|^2 \leqslant \frac{\underline{M}^2 \left[ (\underline{\mathtt{K}})^{\mathtt{m}+1} \right]^2}{[\Gamma(\mathtt{m})]^2} \int_0^1 \underline{\mathscr{W}}(\mathtt{t}_1) d\mathtt{t}_1 \\
&\leqslant \frac{\underline{M}^2 \left[ \underline{\mathtt{K}}^{\mathtt{m}+1} \right]^2}{[\Gamma(\mathtt{m})]^2} \int_0^1 \sqrt{\frac{\mathtt{t}_1}{1-\mathtt{t}_1}} d\mathtt{t}_1 \\
&\leqslant \frac{\underline{M}^2 \left[ \underline{\mathtt{K}}^{\mathtt{m}+1} \right]^2}{[\Gamma(\mathtt{m})]^2} \frac{\pi}{2}.
\end{aligned}$$

Thus, we obtain the bound:

$$\|\underline{\mathtt{U}}(\mathtt{t}_1) - \underline{\mathtt{U}}_{\mathtt{m}}(\mathtt{t}_1)\| \leqslant \frac{\underline{M} \left[ \underline{\mathtt{K}}^{\mathtt{m}+1} \right]}{[\Gamma(\mathtt{m})]} \sqrt{\frac{\pi}{2}}.$$

Similarly, for the upper part, we have:

$$\left\|\overline{\mathsf{U}}(\mathtt{t_1}) - \overline{\mathsf{U}}_{\mathtt{m}}(\mathtt{t_1})\right\| \leqslant \frac{\overline{M}\left[\overline{\mathsf{K}}^{\mathtt{m}+1}\right]}{[\Gamma(\mathtt{m})]}\sqrt{\frac{\pi}{2}},$$

where $\overline{M} = \max_{\nu\in[0,1]} |\overline{\mathsf{U}}^{(\mathtt{m}+1)}(\nu)|$ and $\overline{\mathsf{K}} = \max\{1-\mathtt{t}_0, \mathtt{t}_0\}$. □

**Theorem 3.2.** *Let* $\mathsf{U}(\mathtt{t_1})$ *be fuzzy-valued functions, and let* $\mathsf{U}_{\mathtt{m}}(\mathtt{t_1})$ *represent the best least-squares approximation of* $\mathsf{U}(\mathtt{t_1})$ *as defined in Eq*(3.5)*. The uniform distance between these two functions, denoted by* $\mathbf{D}^*(\mathsf{U}(\mathtt{t_1}), \mathsf{U}_{\mathtt{m}}(\mathtt{t_1}))$*, satisfies the following inequality:*

$$\mathbf{D}^*(\mathsf{U}(\mathtt{t_1}), \mathsf{U}_{\mathtt{m}}(\mathtt{t_1})) \leqslant \frac{M(\mathsf{K})^{\mathtt{m}+1}}{\Gamma(\mathtt{m})}\sqrt{\frac{\pi}{2}}.$$

**Proof.** From Definition 2.4, we express $\mathsf{U}(\mathtt{t_1})$ as:

$$\mathbf{D}^*(\mathsf{U}(\mathtt{t_1}), \mathsf{U}_{\mathtt{m}}(\mathtt{t_1})) = \sup_{\mathtt{t_1}\in[0,1]} D(\mathsf{U}(\mathtt{t_1}), \mathsf{U}_{\mathtt{m}}(\mathtt{t_1}))$$
$$= \sup_{\mathtt{t_1}\in[0,1]}\ \sup_{\Upsilon\in[0,1]} \max\{|\underline{\mathsf{U}}(\mathtt{t_1}) - \underline{\mathsf{U}}_{\mathtt{m}}(\mathtt{t_1})|, |\overline{\mathsf{U}}(\mathtt{t_1}) - \overline{\mathsf{U}}_{\mathtt{m}}(\mathtt{t_1})|\}.$$

Using 3.1, we have

$$\mathbf{D}^*(\mathsf{U}(\mathtt{t_1}), \mathsf{U}_{\mathtt{m}}(\mathtt{t_1})) \leqslant \sup_{\Upsilon\in[0,1]} \max\left\{\frac{\underline{M}\left[\underline{\mathsf{K}}^{\mathtt{m}+1}\right]}{[\Gamma(\mathtt{m})]}\sqrt{\frac{\pi}{2}}, \frac{\overline{M}\left[\overline{\mathsf{K}}^{\mathtt{m}+1}\right]}{[\Gamma(\mathtt{m})]}\sqrt{\frac{\pi}{2}}\right\}.$$

Thus, we obtain the final result:

$$\mathbf{D}^*(\mathsf{U}(\mathtt{t_1}), \mathsf{U}_{\mathtt{m}}(\mathtt{t_1})) \leqslant \frac{M(\mathsf{K})^{\mathtt{m}+1}}{\Gamma(\mathtt{m})}\sqrt{\frac{\pi}{2}}.$$

□

**Theorem 3.3.** *Let* $\mathsf{U}$ *and* $\mathscr{X}$ *be two fuzzy-valued functions. The following inequality holds for their derivative approximations:*

$$\mathbf{D}^*({}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathsf{U}(\mathtt{t_1}), {}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathscr{X}(\mathtt{t_1})) \leqslant \Theta_\Upsilon \frac{M(\mathsf{K})^{\mathtt{m}+1}}{\Gamma(\mathtt{m})}\sqrt{\frac{\pi}{2}},$$

*where* $\Theta_\Upsilon = \max_{\Upsilon\in[0,1]}\{\underline{\Theta}_\Upsilon, \overline{\Theta}_\Upsilon\}$.

**Proof.** Based on Definition 2.4, we can express the following:

$$\mathbf{D}^*({}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathsf{U}(\mathtt{t_1}), {}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathscr{X}(\mathtt{t_1})) = \sup_{\mathtt{t_1}\in[0,1]} \mathbf{D}({}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathsf{U}(\mathtt{t_1}), {}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathscr{X}(\mathtt{t_1})) =$$

$$\sup_{\mathtt{t_1}\in[0,1]}\ \sup_{\Upsilon\in[0,1]} \max\{|{}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\underline{\mathsf{U}}(\mathtt{t_1}) - {}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\underline{\mathscr{X}}(\mathtt{t_1})|, |{}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\overline{\mathsf{U}}(\mathtt{t_1}) - {}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\overline{\mathscr{X}}(\mathtt{t_1})|\}.$$

Using Theorem 2.1, we get:

$$\mathbf{D}^*\left({}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathsf{U}(\mathtt{t_1}), {}^C D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathscr{X}(\mathtt{t_1})\right)$$
$$\leqslant \sup_{\Upsilon\in[0,1]} \max\{\underline{\Theta}_\Upsilon \left\|\underline{\mathsf{U}}(\mathtt{t_1}) - \underline{\mathscr{X}}(\mathtt{t_1})\right\|, \overline{\Theta}_\Upsilon \left\|\overline{\mathsf{U}}(\mathtt{t_1}) - \overline{\mathscr{X}}(\mathtt{t_1})\right\|\}.$$

Next, for the term involving the lower bound of the fuzzy functions, we have:

$$
\begin{aligned}
&|{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathscr{X}}(\mathtt{t}_1)| = |{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\left(\underline{\mathsf{U}}(\mathtt{t}_1) - \underline{\mathscr{X}}(\mathtt{t}_1)\right)| \\
&= \left|\frac{1}{\Gamma(1-\varsigma(\mathtt{t}_1))}\int_0^{\mathtt{t}_1}(\mathtt{t}_1 - \mathrm{s})^{-\varsigma(\mathtt{t}_1)}\left(\underline{\mathsf{U}}(\mathrm{s}) - \underline{\mathscr{X}}(\mathrm{s})\right)' ds\right| \\
&\leqslant \frac{\mathfrak{T}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(1-\varsigma(\mathtt{t}_1))(1-\varsigma(\mathtt{t}_1))}\left|\int_0^{\mathtt{t}_1}\left(\underline{\mathsf{U}}(\mathrm{s}) - \underline{\mathscr{X}}(\mathrm{s})\right)' ds\right| \\
&\leqslant \frac{\mathfrak{T}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(2-\varsigma(\mathtt{t}_1))}\left|\int_0^{\mathtt{t}_1}\left(\underline{\mathsf{U}}(\mathrm{s}) - \underline{\mathscr{X}}(\mathrm{s})\right)' ds\right| \\
&\leqslant \frac{\mathfrak{T}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(2-\varsigma(\mathtt{t}_1))}\left\|\underline{\mathsf{U}}(\mathtt{t}_1) - \underline{\mathscr{X}}(\mathtt{t}_1)\right\| \\
&\leqslant \underline{\Theta}_{\curlyvee}\left\|\underline{\mathsf{U}}(\mathtt{t}_1) - \underline{\mathscr{X}}(\mathtt{t}_1)\right\| \\
&|{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1) - {}^{C}D_{\mathtt{t}_1}^{wp(\mathtt{t}_1)}\underline{\mathscr{X}}(\mathtt{t}_1)| \leqslant \underline{\Theta}_{\curlyvee}\frac{\underline{M}\left[\underline{\mathsf{K}}^{\mathtt{m}+1}\right]}{[\Gamma(\mathtt{m})]}\sqrt{\frac{\pi}{2}}.
\end{aligned}
$$

Similarly, for the upper bound of the fuzzy functions:

$$
|{}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathsf{U}}(\mathtt{t}_1) - {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathscr{X}}(\mathtt{t}_1)| \leqslant \overline{\Theta}_{\curlyvee}\frac{\overline{M}\left[\overline{\mathsf{K}}^{\mathtt{m}+1}\right]}{[\Gamma(\mathtt{m})]}\sqrt{\frac{\pi}{2}}.
$$

By applying the properties of the Hausdorff distance, which satisfies $\mathbf{D}(\mu u, \mu v) = |\mu|\,\mathbf{D}(u, v)$, we obtain:

$$
\begin{aligned}
\mathbf{D}^{*}\left({}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1), {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathscr{X}(\mathtt{t}_1)\right) &\leqslant \sup_{\mathtt{t}_1\in[0,1]}\left(\Theta_{\curlyvee}\mathbf{D}(\mathsf{U}(\mathtt{t}_1), \mathscr{X}(\mathtt{t}_1))\right) \\
&\leqslant \Theta_{\curlyvee}\mathbf{D}^{*}(\mathsf{U}(\mathtt{t}_1), \mathscr{X}(\mathtt{t}_1)).
\end{aligned}
$$

□

## 3.4. Operational matrix of derivative operator using SCP3

To solve the fuzzy variable order fractional differential equation (VO-FFDEs) numerically, we propose transforming the fuzzy variable order fractional Caputo's derivative operators into matrix forms as:

$$
{}_{0}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\vartheta(\mathtt{t}_1) = {}_{0}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\left(\mathsf{P}\mathtt{T}(\mathtt{t}_1)\right) = \mathsf{P}\left({}_{0}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathtt{T}(\mathtt{t}_1)\right).
$$

From equation (2.1), we obtain

$$
{}_{0}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\vartheta(\mathtt{t}_1) = \mathsf{P}\left[0 \;\; \tfrac{\Gamma(2)}{\Gamma(2-\varsigma(\mathtt{t}_1))}{\mathtt{t}_1}^{1-\varsigma(\mathtt{t}_1)} \;\; \dots \;\; \tfrac{\Gamma(\mathtt{m}+1)}{\Gamma(\mathtt{m}+1-\varsigma(\mathtt{t}_1))}{\mathtt{t}_1}^{\mathtt{m}-\varsigma(\mathtt{t}_1)}\right]^{T}
$$

$$
= \mathsf{P}\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \tfrac{\Gamma(2)}{\Gamma(2-\varsigma(\mathtt{t}_1))}{\mathtt{t}_1}^{-\varsigma(\mathtt{t}_1)} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \tfrac{\Gamma(\mathtt{m}+1)}{\Gamma(\mathtt{m}+1-\varsigma(\mathtt{t}_1))}{\mathtt{t}_1}^{-\varsigma(\mathtt{t}_1)} \end{bmatrix}\begin{bmatrix} 1 \\ \mathtt{t}_1 \\ \vdots \\ {\mathtt{t}_1}^{\mathtt{m}} \end{bmatrix}.
$$

By using Eq (3.4), we have

$$ {}_0^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)} \vartheta(\mathtt{t}_1) = \mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1), \tag{3.7}$$

where

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\varsigma(\mathtt{t}_1))}\mathtt{t}_1{}^{-\varsigma(\mathtt{t}_1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{\Gamma(\mathtt{m}+1)}{\Gamma(\mathtt{m}+1-\varsigma(\mathtt{t}_1))}\mathtt{t}_1{}^{-\varsigma(\mathtt{t}_1)} \end{bmatrix}.$$

The expression $\mathsf{PMP}^{-1}$ is referred to as the operational matrix for the fuzzy variable order derivative by SCP3.

Consequently, the operational matrix for ${}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1)$ can be formulated as:

$$\begin{aligned} {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1) &\cong {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\left(\mathsf{C}^T\vartheta(\mathtt{t}_1)\right) = \mathsf{C}^T\left({}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\vartheta(\mathtt{t}_1)\right) \\ &= \mathsf{C}^T\mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1). \end{aligned}$$

From Definition 2.10, we approximate the fuzzy ${}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1)$ as:

$$\begin{aligned} {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\aleph(\mathtt{t}_1,\Upsilon) &= \left[{}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\aleph}(\mathtt{t}_1,\Upsilon), {}^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\aleph}(\mathtt{t}_1,\Upsilon)\right] \\ &= \left[\underline{\mathsf{C}}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1), \overline{\mathsf{C}}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1)\right]. \end{aligned} \tag{3.8}$$

# 4. Proposed method

To obtain the solution $\mathsf{U}(\mathtt{t}_1)$, we propose an algorithm comprising the following steps:

**Step 1**.Start with Eq.(1.1).

**Step 2**.Approximate the unknown function $\mathsf{U}(\mathtt{t})$ and its derivative by using Eqs.(3.6) and (3.8), then substitute these approximations back into Eq.(1.1).

**Step 3**.Compute the operational matrices and substitute in Eq.(1.1).

The results from **Step 2** and **Step 3** are as follows:

$$\begin{cases} \mathsf{C}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1) = f(\mathtt{t}_1, \mathsf{C}_{\Upsilon}^T\vartheta(\mathtt{t}_1)), \\ \\ \mathsf{C}_{\Upsilon}^T\vartheta(0) = \mathsf{U}_0. \end{cases} \tag{4.1}$$

Convert the problems Eq.(4.1) to the corresponding systems of boundary value problems as:

$$\begin{cases} \underline{\mathsf{C}}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1) = f(\mathtt{t}_1, \underline{\mathsf{C}}_{\Upsilon}^T\vartheta(\mathtt{t}_1), \\ \\ \mathsf{C}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(0) = \underline{\mathsf{U}}_0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \overline{\mathsf{C}}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(\mathtt{t}_1) = f(\mathtt{t}_1, \overline{\mathsf{C}}_{\Upsilon}^T\vartheta(\mathtt{t}_1)), \\ \\ \overline{\mathsf{C}}_{\Upsilon}^T\mathsf{PMP}^{-1}\vartheta(0) = \overline{\mathsf{U}}_0. \end{cases} \tag{4.3}$$

**Step 4**. To determine the unknown coefficients $\mathsf{C}_{\curlyvee}$, we apply collocation points $\mathtt{t_1}_r = \frac{(2r+1)}{(2\mathtt{m}+2)}, r = 0, 1, \ldots, \mathtt{m}$. By substituting these points into Eqs. (4.2) and (4.3). we obtain a system of algebraic equations. Solving this system provides the unknown fuzzy coefficient vectors $\mathsf{C}_{\curlyvee}$. Finally, the numerical solution is obtained as $\mathsf{U}(\mathtt{t_1}, \curlyvee) = \left[\underline{\mathsf{C}}_{\curlyvee}^T \vartheta(\mathtt{t_1}), \overline{\mathsf{C}}_{\curlyvee}^T \vartheta(\mathtt{t_1})\right]$.

## 5. Numerical examples

Several numerical examples are presented in this section to support the theoretical discussion and illustrate the accuracy of the suggested approach.

**Example 5.1.** Consider the VO-FFDEs

$$ {}^{C}D_{\mathtt{t_1}}^{\varsigma(\mathtt{t_1})}\mathsf{U}(\mathtt{t_1}) = g(\mathtt{t_1}), \tag{5.1}$$

with fuzzy number initial condition

$$\mathsf{U}(0, \curlyvee) = \left[\underline{\mathsf{U}}_0, \overline{\mathsf{U}}_0\right]. \tag{5.2}$$

Suppose $\mathsf{U}(\mathtt{t_1})$ is (i)-differentiable. Then Eqs. (5.1) and (5.2) are written as the following system:

$$\begin{cases} {}^{C}D_{t}^{\varsigma(\mathtt{t_1})}\underline{\mathsf{U}}(\mathtt{t_1}, \curlyvee) = (1+\curlyvee)\left(2\frac{\mathtt{t_1}^{2-\varsigma(\mathtt{t_1})}}{\Gamma(3-\varsigma(\mathtt{t_1}))} + 3\frac{\mathtt{t_1}^{1-\varsigma(\mathtt{t_1})}}{\Gamma(2-\varsigma(\mathtt{t_1}))}\right), \\ \underline{\mathsf{U}}(0, \curlyvee) = \underline{\mathsf{U}}_0 \in \mathbb{E}_{\mathrm{F}}, \end{cases}$$

and

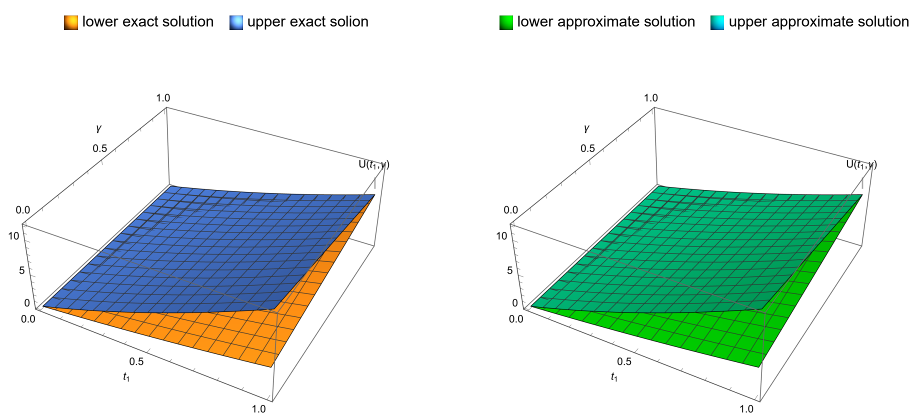
$$\begin{cases} {}^{C}D_{t}^{\varsigma(\mathtt{t_1})}\overline{\mathsf{U}}(\mathtt{t_1}, \curlyvee) = (3-\curlyvee)\left(2\frac{\mathtt{t_1}^{2-\varsigma(\mathtt{t_1})}}{\Gamma(3-\varsigma(\mathtt{t_1}))} + 3\frac{\mathtt{t_1}^{1-\varsigma(\mathtt{t_1})}}{\Gamma(2-\varsigma(\mathtt{t_1}))}\right), \\ \overline{\mathsf{U}}(0, \curlyvee) = \overline{\mathsf{U}}_0 \in \mathbb{E}_{\mathrm{F}}, \end{cases}$$

where $\left[\underline{\mathsf{U}}_0, \overline{\mathsf{U}}_0\right] = [1+\curlyvee, 3-\curlyvee]$, and $\varsigma(\mathtt{t_1}) = 1 - e^{-2\mathtt{t_1}}$. The exact solution is $\left[\underline{\mathsf{U}}(\mathtt{t_1}, \curlyvee), \overline{\mathsf{U}}(\mathtt{t_1}, \curlyvee)\right] = \left[(1+\curlyvee)(\mathtt{t_1}^2 + 3\mathtt{t_1}), (3-\curlyvee)(\mathtt{t_1}^2 + 3\mathtt{t_1})\right]$.

We addressed the problem by applying the proposed method, with numerical results illustrated in Figures 1, 2, 3, and Table 1. In Figure 1, the exact and approximate solutions, obtained by setting $\mathtt{m} = 4$, are presented. Table 1 provides the absolute errors (AE) corresponding to various values of $\mathtt{t_1}$ and $\curlyvee$ when $\mathtt{m} = 4$. Figure 2 provides further details of the solutions and their accuracy: Figures 2(a) and 2(c) display the lower and upper absolute errors when $\mathtt{t_1} = 0.1$ and for various values of $\curlyvee$, and Figures 2(b) and 2(d) present the lower and upper absolute errors for $\mathtt{t_1} = 0.9$ across different values of $\curlyvee$. Figure 3 shows the lower and upper approximate solutions across different values of $\curlyvee$. These graphical and tabulated results highlight the effectiveness and precision of the proposed method.

**Table 1.** Absolute Error

| $(t_1, \Upsilon)$ | $\underline{U}(t_1, \Upsilon)$ | $\overline{U}(t_1, \Upsilon)$ | Lower Absolute Error | Upper Absolute Error |
|---|---|---|---|---|
| (0.1,0.1) | 0.341 | 0.899 | $5.55112\times10^{-17}$ | $9.99201\times10^{-16}$ |
| (0.2,0.2) | 0.768 | 1.792 | 0 | $6.66134\times10^{-16}$ |
| (0.3,0.3) | 1.287 | 2.673 | $4.44089\times10^{-16}$ | $4.44089\times10^{-16}$ |
| (0.4,0.4) | 1.904 | 3.536 | 0 | 0 |
| (0.5,0.5) | 2.625 | 4.375 | $4.44089\times10^{-16}$ | 0 |
| (0.6,0.6) | 3.456 | 5.184 | $8.88178\times10^{-16}$ | 0 |
| (0.7,0.7) | 4.403 | 5.957 | $8.88178\times10^{-16}$ | $1.77636\times10^{-15}$ |
| (0.8,0.8) | 5.472 | 6.688 | 0 | $8.88178\times10^{-16}$ |
| (0.9,0.9) | 6.669 | 7.371 | $8.88178\times10^{-16}$ | $1.77636\times10^{-15}$ |




(a) Lower and upper exact solutions

(b) Lower and upper approximate solutions

**Figure 1.** (a) Exact and (b) approximate lower and upper solutions when $m = 4$ for Example 5.1.

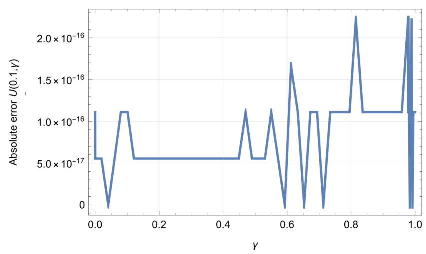
**Example 5.2.** Consider the VO-FFDEs

$$^{C}D_{t_1}^{\varsigma(t_1)}U(t_1) - U(t_1) = g(t_1), \tag{5.3}$$

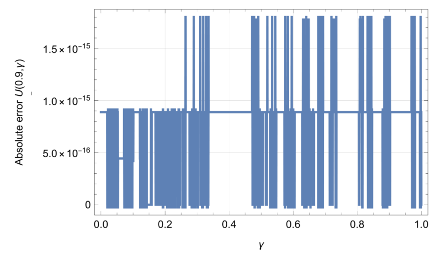
with fuzzy number initial condition

$$U(0, \Upsilon) = \left[\underline{U}_0, \overline{U}_0\right]. \tag{5.4}$$

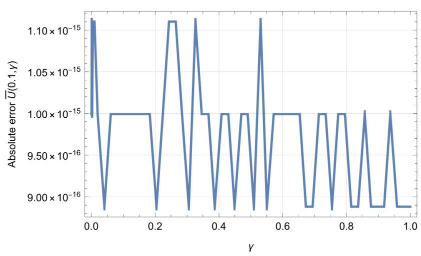
Suppose $U(t_1)$ is (i)-differentiable. Then Eqs.(5.3) and (5.4) are written as the



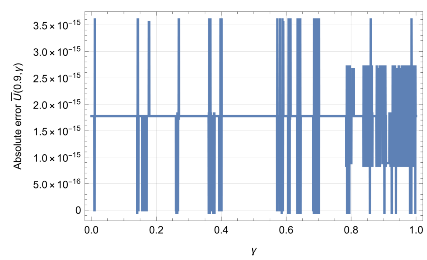

(a) $\mathtt{t}_1 = 0.1$




(b) $\mathtt{t}_1 = 0.9$




(c) $\mathtt{t}_1 = 0.1$




(d) $\mathtt{t}_1 = 0.9$

**Figure 2.** The corresponding absolute errors for $\underline{\mathsf{U}}(\mathtt{t}_1, \curlyvee)$ and $\overline{\mathsf{U}}(\mathtt{t}_1, \curlyvee)$ when $\mathtt{m} = 4$, for Example 5.1.

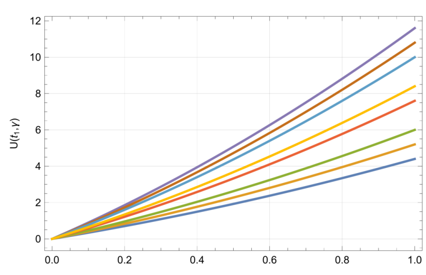



**Figure 3.** Approximate solutions $\underline{\mathsf{U}}(\mathtt{t}_1, \curlyvee)$ and $\overline{\mathsf{U}}(\mathtt{t}_1, \curlyvee)$ for various values of $\curlyvee$ , for Example 5.1.

following system:

$$\begin{cases} {}^{C}D^{\varsigma(\mathtt{t_1})}_{\mathtt{t_1}}\underline{\mathsf{U}}(\mathtt{t_1},\Upsilon)-\underline{\mathsf{U}}(\mathtt{t_1},\Upsilon)=(-1+\Upsilon)\left(\frac{\mathtt{t_1}^{2-\varsigma(\mathtt{t_1})}}{\Gamma(3-\varsigma(\mathtt{t_1}))}-\frac{1}{2}\mathtt{t_1}^2\right), \\ \underline{\mathsf{U}}(0,\Upsilon)=\underline{\mathsf{U}}_0\in\mathbb{E}_{\mathrm{F}}. \end{cases}$$

And

$$\begin{cases} {}^{C}D^{\varsigma(\mathtt{t_1})}_{\mathtt{t_1}}\overline{\mathsf{U}}(\mathtt{t_1},\Upsilon)-\overline{\mathsf{U}}(\mathtt{t_1},\Upsilon)=(1-\Upsilon)\left(\frac{\mathtt{t_1}^{2-\varsigma(\mathtt{t_1})}}{\Gamma(3-\varsigma(\mathtt{t_1}))}-\frac{1}{2}\mathtt{t_1}^2\right), \\ \overline{\mathsf{U}}(0,\Upsilon)=\overline{\mathsf{U}}_0\in\mathbb{E}_{\mathrm{F}}, \end{cases}$$

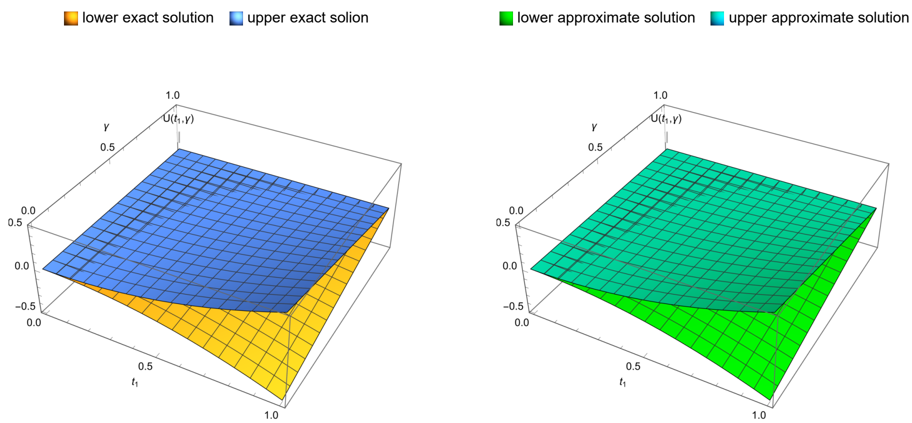
where $\left[\underline{\mathsf{U}}_0,\overline{\mathsf{U}}_0\right]=[-1+\Upsilon,1-\Upsilon]$, and $\varsigma(\mathtt{t_1})=1-0.5e^{-\mathtt{t_1}}$.

The exact solution is $\left[\underline{\mathsf{U}}(\mathtt{t_1},\Upsilon),\overline{\mathsf{U}}(\mathtt{t_1},\Upsilon)\right]=\left[(-1+\Upsilon)\frac{\mathtt{t_1}^2}{2},(1-\Upsilon)\frac{\mathtt{t_1}^2}{2}\right]$.

We addressed the problem by applying the proposed method, with numerical results illustrated in Figures 4, 5, 6, and Table 2. In Figure 4, the exact and approximate solutions, obtained by setting $\mathtt{m}=4$, are presented. Table 2 provides the AE corresponding to various values of $\mathtt{t_1}$ and $\Upsilon$ when $\mathtt{m}=4$, and it includes a comparison with the absolute errors of a similar numerical method using Mittag-Leffler kernels from [27]. Figure 5 shows the lower and upper approximate solutions across different values of $\Upsilon$. Figure 6 provides further details of the solutions and their accuracy: Figures 6(a) and 6(c) display the lower and upper absolute errors when $\mathtt{t_1}=0.1$ and for various values of $\Upsilon$, and Figures 6(b) and 6(d) present the lower and upper absolute errors for $\mathtt{t_1}=0.9$ across different values of $\Upsilon$. These graphical and tabulated results highlight the effectiveness and precision of the proposed method.

**Table 2.** Absolute Error

| $(\mathtt{t_1},\Upsilon)$ | $\underline{\mathsf{U}}(\mathtt{t_1},\Upsilon)$ | $\overline{\mathsf{U}}(\mathtt{t_1},\Upsilon)$ | Lower Absolute Error | Upper Absolute Error |
|---|---|---|---|---|
| (0.1,0.1) | -0.0045 | 0.0045 | $1.02349\times10^{-16}$ | $7.45931\times10^{-17}$ |
| (0.2,0.2) | -0.016 | 0.016 | $2.42861\times10^{-17}$ | $3.46945\times10^{-18}$ |
| (0.3,0.3) | -0.0315 | 0.0315 | $2.08167\times10^{-17}$ | $4.85723\times10^{-17}$ |
| (0.4,0.4) | -0.048 | 0.048 | $3.46945\times10^{-17}$ | $6.245\times10^{-17}$ |
| (0.5,0.5) | -0.0625 | 0.0625 | $2.77556\times10^{-17}$ | $5.55112\times10^{-17}$ |
| (0.6,0.6) | -0.072 | 0.072 | $1.38778\times10^{-17}$ | $4.16334\times10^{-17}$ |
| (0.7,0.7) | -0.0735 | 0.0735 | $1.38778\times10^{-17}$ | $1.38778\times10^{-17}$ |
| (0.8,0.8) | -0.064 | 0.064 | 0 | 0 |
| (0.9,0.9) | -0.0405 | 0.0405 | $1.38778\times10^{-17}$ | $1.38778\times10^{-17}$ |




(a) Lower and upper exact solutions (b) Lower and upper approximate solutions

**Figure 4.** (a) Exact and (b) approximate lower and upper solutions when $\mathtt{m} = 4$ for Example 5.2.

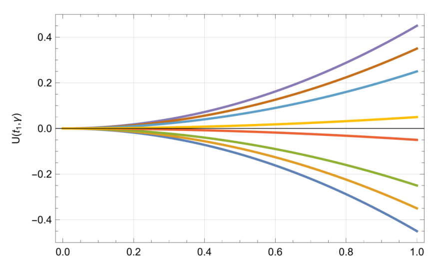
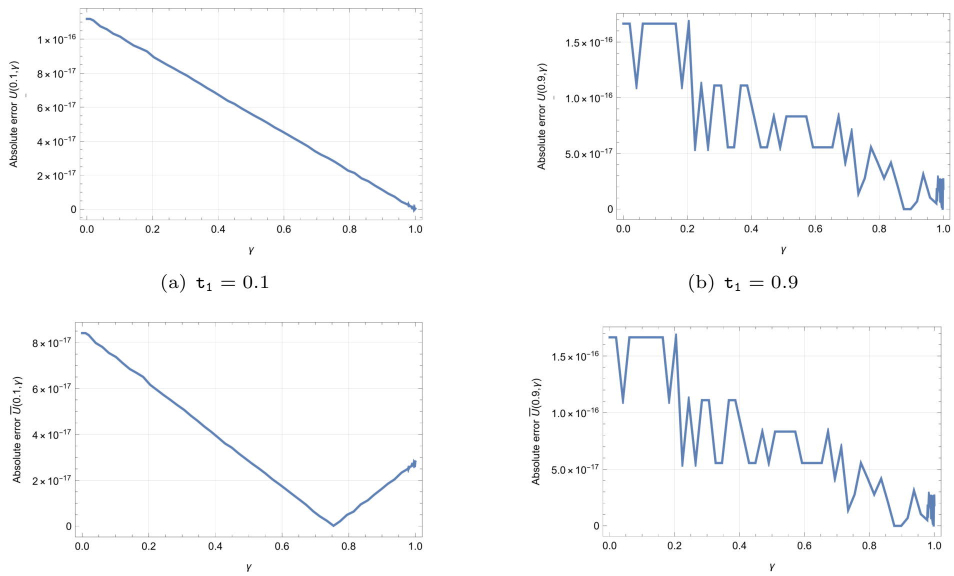



**Figure 5.** Approximate solutions $\underline{\mathrm{U}}(\mathtt{t}_1, \curlyvee)$ and $\overline{\mathrm{U}}(\mathtt{t}_1, \curlyvee)$ for various values of $\curlyvee$, for Example 5.2.




(a) $\mathtt{t}_1 = 0.1$ (b) $\mathtt{t}_1 = 0.9$

(c) $\mathtt{t}_1 = 0.1$ (d) $\mathtt{t}_1 = 0.9$

**Figure 6.** The corresponding absolute errors for $\underline{\mathsf{U}}(\mathtt{t}_1, \Upsilon)$ and $\overline{\mathsf{U}}(\mathtt{t}_1, \Upsilon)$ when $\mathtt{m} = 4$, for Example 5.2.

**Example 5.3.** Consider the VO-FFDEs

$$ {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1) - 10\mathsf{U}'(\mathtt{t}_1) + \mathsf{U}(\mathtt{t}_1) = g(\mathtt{t}_1), \tag{5.5} $$

with fuzzy number initial condition

$$ \mathsf{U}(0, \Upsilon) = \left[\underline{\mathsf{U}}_0, \overline{\mathsf{U}}_0\right]. \tag{5.6} $$

Suppose $\mathsf{U}(\mathtt{t}_1)$ is (i)-differentiable. Then Eqs.(5.3) and (5.4) are written as the following system:

$$ \begin{cases} {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1, \Upsilon) - 10\underline{\mathsf{U}}'(\mathtt{t}_1, \Upsilon) + \underline{\mathsf{U}}(\mathtt{t}_1, \Upsilon) \\ = \Upsilon \left(10 \left(\frac{{\mathtt{t}_1}^{2-\varsigma(\mathtt{t}_1)}}{\Gamma(3-\varsigma(\mathtt{t}_1))} + \frac{{\mathtt{t}_1}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(2-\varsigma(\mathtt{t}_1))}\right) + 5{\mathtt{t}_1}^2 - 90\mathtt{t}_1 - 95\right), \\ \underline{\mathsf{U}}(0, \Upsilon) = \underline{\mathsf{U}}_0 \in \mathbb{E}_{\mathrm{F}}. \end{cases} $$

And

$$ \begin{cases} {}^{C}D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathsf{U}}(\mathtt{t}_1, \Upsilon) - 10\overline{\mathsf{U}}'(\mathtt{t}_1, \Upsilon) + \overline{\mathsf{U}}(\mathtt{t}_1, \Upsilon) \\ = (2 - \Upsilon) \left(10 \left(\frac{{\mathtt{t}_1}^{2-\varsigma(\mathtt{t}_1)}}{\Gamma(3-\varsigma(\mathtt{t}_1))} + \frac{{\mathtt{t}_1}^{1-\varsigma(\mathtt{t}_1)}}{\Gamma(2-\varsigma(\mathtt{t}_1))}\right) + 5{\mathtt{t}_1}^2 - 90\mathtt{t}_1 - 95\right), \\ \overline{\mathsf{U}}(0, \Upsilon) = \overline{\mathsf{U}}_0 \in \mathbb{E}_{\mathrm{F}}, \end{cases} $$

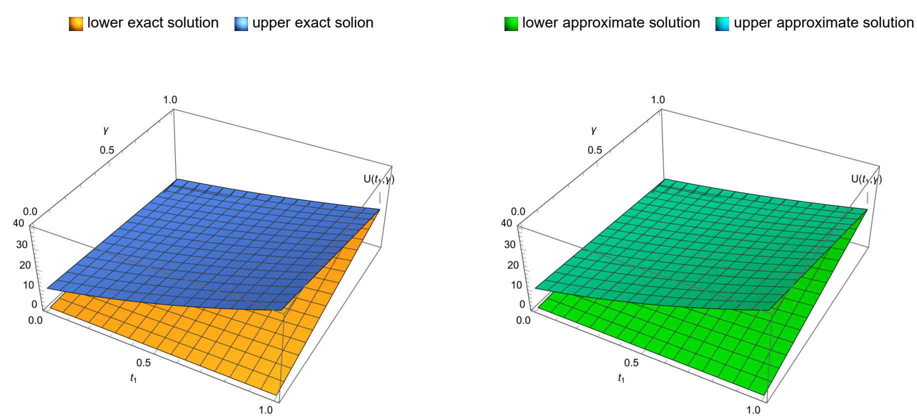
where $\left[\underline{\mathsf{U}}_0, \overline{\mathsf{U}}_0\right] = [\Upsilon, 2 - \Upsilon]$, and $\varsigma(\mathtt{t}_1) = \sin(\mathtt{t}_1)$. The exact solution is

$$ \left[\underline{\mathsf{U}}(\mathtt{t}_1, \Upsilon), \overline{\mathsf{U}}(\mathtt{t}_1, \Upsilon)\right] = \left[\Upsilon(5 + 10\mathtt{t}_1 + 5{\mathtt{t}_1}^2), (2 - \Upsilon)(5 + 10\mathtt{t}_1 + 5{\mathtt{t}_1}^2)\right]. $$

We addressed the problem by applying the proposed SCP3 based spectral method, with numerical results illustrated in Figures 7, 8, and Table 3. Figure 7 compares the exact and approximate solutions when $\mathtt{m} = 4$, showing a very good match. Table 3 lists the absolute errors for different values of $\mathtt{t_1}$ and $\Upsilon$, and Figure 8 shows the lower and upper absolute errors across the domain. These results clearly show that our method is accurate, reliable, and works well for solving fuzzy variable-order differential equations.

**Table 3.** Absolute Error

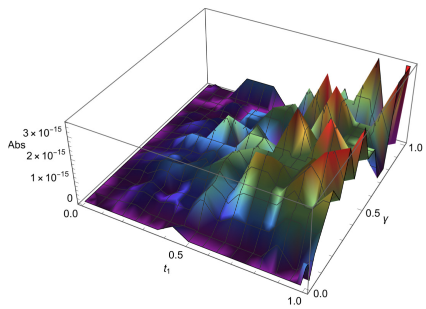
| $(\mathtt{t_1}, \Upsilon)$ | $\underline{\mathsf{U}}(\mathtt{t_1}, \Upsilon)$ | $\overline{\mathsf{U}}(\mathtt{t_1}, \Upsilon)$ | Lower Absolute Error | Upper Absolute Error |
|---|---|---|---|---|
| (0.1,0.1) | 0.605 | 11.495 | 0 | $1.77636\times10^{-15}$ |
| (0.2,0.2) | 1.44 | 12.96 | $4.44089\times10^{-16}$ | $3.55271\times10^{-15}$ |
| (0.3,0.3) | 2.535 | 14.365 | $4.44089\times10^{-16}$ | $3.55271\times10^{-15}$ |
| (0.4,0.4) | 3.92 | 15.68 | $1.33227\times10^{-15}$ | $3.55271\times10^{-15}$ |
| (0.5,0.5) | 5.625 | 16.875 | $1.77636\times10^{-15}$ | $7.105432\times10^{-15}$ |
| (0.6,0.6) | 7.68 | 17.92 | $1.77636\times10^{-15}$ | $3.55271\times10^{-15}$ |
| (0.7,0.7) | 10.115 | 18.785 | $1.77636\times10^{-15}$ | $3.55271\times10^{-15}$ |
| (0.8,0.8) | 12.96 | 19.44 | 0 | 0 |
| (0.9,0.9) | 16.245 | 19.855 | $3.55271\times10^{-15}$ | $3.55271\times10^{-15}$ |



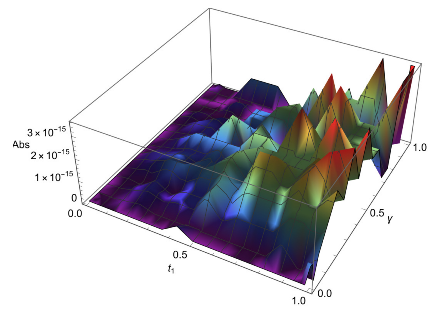

(a) Lower and upper exact solutions

(b) Lower and upper approximate solutions

**Figure 7.** (a) Exact and (b) approximate lower and upper solutions when $\mathtt{m} = 4$ for Example 5.3.




(a) Lower Absolute Error




(b) Upper Absolute Error

**Figure 8.** (a) lower and (b) Upper Absolute Error when $\mathtt{m}=4$ Example 5.3.

**Example 5.4.** Consider the VO-FFDEs

$$\begin{aligned}&\mathtt{p}(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1)+\mathtt{q}_1(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_1(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1)+\mathtt{q}_2(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_2(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1)\\&+\mathtt{q}_3(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_3(\mathtt{t}_1)}\mathsf{U}(\mathtt{t}_1)+\mathtt{r}(\mathtt{t}_1)\mathsf{U}(\mathtt{t}_1)=g(\mathtt{t}_1),\end{aligned}\tag{5.7}$$

with fuzzy number initial condition

$$\mathsf{U}(0,\Upsilon)=\left[\underline{\mathsf{U}}_0,\overline{\mathsf{U}}_0\right],\tag{5.8}$$

and $\mathtt{p}(\mathtt{t}_1)=1,\mathtt{q}_1(\mathtt{t}_1)=\mathtt{t}_1{}^{\frac{1}{2}},\mathtt{q}_2(\mathtt{t}_1)=\mathtt{t}_1{}^{\frac{1}{3}},\mathtt{q}_3(\mathtt{t}_1)=\mathtt{t}_1{}^{\frac{1}{4}},\mathtt{r}(\mathtt{t}_1)=\mathtt{t}_1{}^{\frac{1}{5}},\varsigma(\mathtt{t}_1)=\frac{\mathtt{t}_1}{2},\boldsymbol{\beta}_1(\mathtt{t}_1)=\frac{\mathtt{t}_1}{3},\boldsymbol{\beta}_2(\mathtt{t}_1)=\frac{\mathtt{t}_1}{4},\boldsymbol{\beta}_3(\mathtt{t}_1)=\frac{\mathtt{t}_1}{5}$.

Suppose $\mathsf{U}(\mathtt{t}_1)$ is (i)-differentiable. Then Eqs.(5.7) and (5.8) are written as the following system:

$$\begin{cases}\mathtt{p}(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)+\mathtt{q}_1(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_1(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)+\mathtt{q}_2(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_2(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)\\+\mathtt{q}_3(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_3(\mathtt{t}_1)}\underline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)+\mathtt{r}(\mathtt{t}_1)\underline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)=\underline{g}(\mathtt{t}_1,\Upsilon),\\\underline{\mathsf{U}}(0,\Upsilon)=\underline{\mathsf{U}}_0\in\mathbb{E}_{\mathrm{F}}.\end{cases}$$

And

$$\begin{cases}\mathtt{p}(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)}\overline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)+\mathtt{q}_1(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_1(\mathtt{t}_1)}\overline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)+\mathtt{q}_2(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_2(\mathtt{t}_1)}\overline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)\\+\mathtt{q}_3(\mathtt{t}_1)^C D_{\mathtt{t}_1}^{\boldsymbol{\beta}_3(\mathtt{t}_1)}\overline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)+\mathtt{r}(\mathtt{t}_1)\overline{\mathsf{U}}(\mathtt{t}_1,\Upsilon)=\overline{g}(\mathtt{t}_1,\Upsilon),\\\overline{\mathsf{U}}(0,\Upsilon)=\overline{\mathsf{U}}_0\in\mathbb{E}_{\mathrm{F}},\end{cases}$$

where $\left[\underline{\mathsf{U}}_0,\overline{\mathsf{U}}_0\right]=[1+\Upsilon,2-\Upsilon]$, and $g(\mathtt{t}_1)=\left[\underline{g}(\mathtt{t}_1,\Upsilon),\overline{g}(\mathtt{t}_1,\Upsilon)\right]$. Here

$$\begin{aligned}\underline{g}(\mathtt{t}_1,\Upsilon)=(1+\Upsilon)\Big(&-\mathtt{p}(\mathtt{t}_1)\frac{\mathtt{t}_1{}^{2-\varsigma(\mathtt{t}_1)}}{\Gamma(3-\varsigma(\mathtt{t}_1))}-\mathtt{q}_1(\mathtt{t}_1)\frac{\mathtt{t}_1{}^{2-\boldsymbol{\beta}_1(\mathtt{t}_1)}}{\Gamma(3-\boldsymbol{\beta}_1(\mathtt{t}_1))}\\&-\mathtt{q}_2(\mathtt{t}_1)\frac{\mathtt{t}_1{}^{2-\boldsymbol{\beta}_2(\mathtt{t}_1)}}{\Gamma(3-\boldsymbol{\beta}_2(\mathtt{t}_1))}-\mathtt{q}_3(\mathtt{t}_1)\frac{\mathtt{t}_1{}^{2-\boldsymbol{\beta}_3(\mathtt{t}_1)}}{\Gamma(3-\boldsymbol{\beta}_3(\mathtt{t}_1))}+\mathtt{r}(\mathtt{t}_1)(2-\frac{\mathtt{t}_1{}^2}{2})\Big),\end{aligned}$$

and

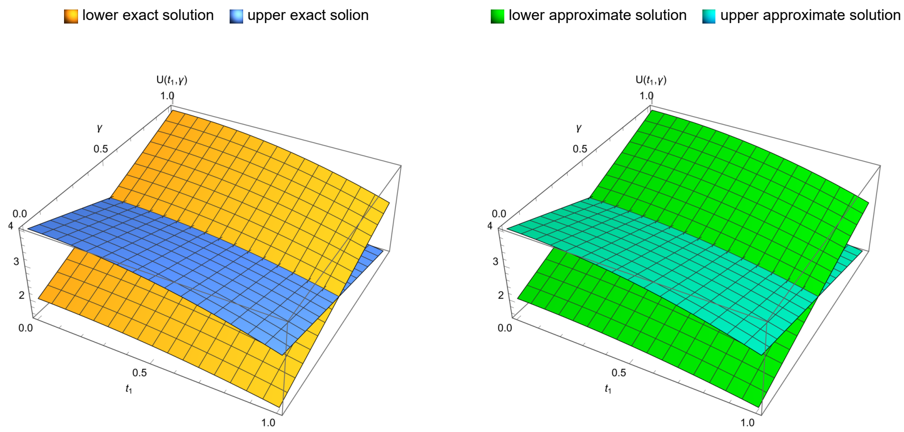
$$\overline{g}(\mathtt{t}_1, \Upsilon) = (2-\Upsilon)\Big( -\mathtt{p}(\mathtt{t}_1)\frac{{\mathtt{t}_1}^{2-\varsigma(\mathtt{t}_1)}}{\Gamma(3-\varsigma(\mathtt{t}_1))} - \mathtt{q}_1(\mathtt{t}_1)\frac{{\mathtt{t}_1}^{2-\boldsymbol{\beta}_1(\mathtt{t}_1)}}{\Gamma(3-\boldsymbol{\beta}_1(\mathtt{t}_1))} - \mathtt{q}_2(\mathtt{t}_1)\frac{{\mathtt{t}_1}^{2-\boldsymbol{\beta}_2(\mathtt{t}_1)}}{\Gamma(3-\boldsymbol{\beta}_2(\mathtt{t}_1))} - \mathtt{q}_3(\mathtt{t}_1)\frac{{\mathtt{t}_1}^{2-\boldsymbol{\beta}_3(\mathtt{t}_1)}}{\Gamma(3-\boldsymbol{\beta}_3(\mathtt{t}_1))} + \mathtt{r}(\mathtt{t}_1)(2-\frac{{\mathtt{t}_1}^2}{2})\Big).$$

The exact solution is $\big[\underline{\mathsf{U}}(\mathtt{t}_1, \Upsilon), \overline{\mathsf{U}}(\mathtt{t}_1, \Upsilon)\big] = \Big[(1+\Upsilon)(2-\frac{{\mathtt{t}_1}^2}{2}), (2-\Upsilon)(2-\frac{{\mathtt{t}_1}^2}{2})\Big]$.

We addressed the problem by applying the proposed SCP3 based spectral method, with numerical results presented in Figures 9, 10, and Table 4. Figure 9 shows a strong agreement between the exact and approximate solutions for $\mathtt{m} = 2$. Table 4 reports the absolute errors for various values of $\mathtt{t}_1$ and $\Upsilon$, confirming the high accuracy of the method. Additionally, Figure 9 illustrates the lower and upper absolute errors across the domain.

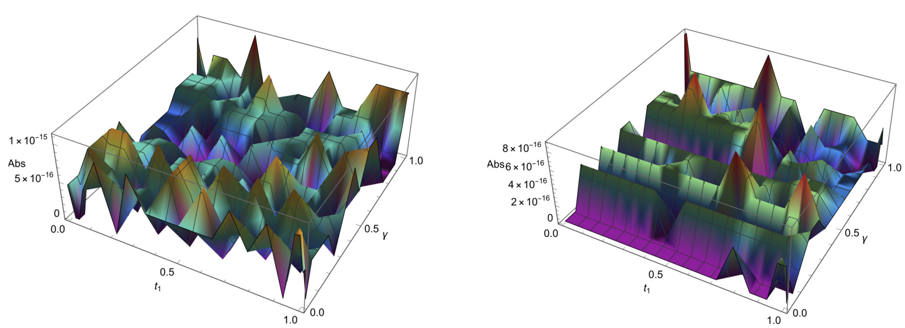
**Table 4.** Absolute Error

| $(\mathtt{t}_1, \Upsilon)$ | $\underline{\mathsf{U}}(\mathtt{t}_1, \Upsilon)$ | $\overline{\mathsf{U}}(\mathtt{t}_1, \Upsilon)$ | Lower Absolute Error | Upper Absolute Error |
|---|---|---|---|---|
| (0.1,0.1) | 2.1945 | 3.7905 | $4.44089\times10^{-16}$ | $4.44089\times10^{-16}$ |
| (0.2,0.2) | 2.376 | 3.564 | 0 | 0 |
| (0.3,0.3) | 2.5415 | 3.3235 | 0 | $4.44089\times10^{-16}$ |
| (0.4,0.4) | 2.688 | 3.072 | $4.44089\times10^{-16}$ | 0 |
| (0.5,0.5) | 2.8125 | 2.8125 | 0 | $4.44089\times10^{-16}$ |
| (0.6,0.6) | 2.912 | 2.548 | $4.44089\times10^{-16}$ | $4.44089\times10^{-16}$ |
| (0.7,0.7) | 2.9835 | 2.2815 | $8.88178\times10^{-16}$ | 0 |
| (0.8,0.8) | 3.024 | 2.016 | 0 | 0 |
| (0.9,0.9) | 3.0305 | 1.7545 | 0 | $4.44089\times10^{-16}$ |




(a) lower and Upper Exact Solutions (b) lower and Upper Approximate Solutions

**Figure 9.** (a) Exact and (b) approximate lower and Upper Solutions when $\mathtt{m} = 2$ Example 5.4.




(a) lower Absolute Error (b) Upper Absolute Error

**Figure 10.** (a)lower and (b) Upper Absolute Error when $\mathtt{m} = 2$ Example 5.4.

# 6. Conclusion

In conclusion, this study presents an effective numerical technique for solving VO-FFDEs under fuzzy initial conditions using the Caputo derivative. By employing OMs of SCP3 and utilizing the $\Upsilon$-cut representation of fuzzy-valued functions, the original FFDE problem is transformed into a system of nonlinear algebraic equations.The numerical solution is then determined by solving this system of equations to ascertain the coefficients denoted as $\mathsf{C}^T$. The accuracy and effectiveness of the proposed method are validated through numerical experiments. As shown in Tables 1 and 2, the method achieves highly accurate results, with errors approaching machine precision (on the order of $10^{-17}$). Furthermore, the results in Tables 3 and 4 confirm the robustness of the approach for multi-term and extended cases, reinforcing its stability and precision under more complex configurations. This approach is efficient, straightforward to implement, and adaptable for variable-order differential equations with fuzzy elements, offering a powerful tool for obtaining precise solutions in complex differential problems.

## Acknowledgements

The authors thank SRTM University, Taiz University, and the journal team for their support and assistance.

## References

[1] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[2] W. Chen, *A speculative study of 2/3-order fractional Laplacian modeling of turbulence: Some thoughts and conjectures*, Chaos, 2006, 16(2).

[3] W. Chen, H. Sun, X. Zhang and D. Korošak, *Anomalous diffusion modeling by fractal and fractional derivatives*, Computers & Mathematics with Applications, 2010, 59(5), 1754–1758.

[4] W.F. Ahmed, D.D. Pawar and W.D. Patil, *Solution of fractional kinetic equations involving Laguerre polynomials via Sumudu transform*, Journal of Function Spaces, 2024, 2024(1), 9935562.

[5] A.Y. Salamooni and D.D. Pawar, *Existence and uniqueness of nonlocal boundary conditions for Hilfer-Hadamard-type fractional differential equations*, Advances in Difference Equations, 2021, 2021, 1–20.

[6] S.T. Thabet, W.F. Ahmed and D.D. Pawar, *On fractional kinetic equation relating to the generalized k-Bessel function by the Mellin transform*, Journal of Fractional Calculus and Nonlinear Systems, 2022, 3(2), 1–2.

[7] E.H. Doha, M.A. Abdelkawy, A.Z. Amin and D. Baleanu, *Spectral technique for solving variable-order fractional Volterra integro-differential equations*, Numerical Methods for Partial Differential Equations, 2018, 34(5), 1659–1677.

[8] X.J. Yang, *General fractional derivatives: Theory, Methods and Applications*, Chapman and Hall/CRC, 2019.

[9] C.M. Soon, C.F. Coimbra and M.H. Kobayashi, *The variable viscoelasticity oscillator*, Annalen der Physik, 2005, 517(6), 378–389.

[10] R.M. Ganji, H. Jafari and D. Baleanu, *A new approach for solving multi variable orders differential equations with Mittag-Leffler kernel*, Chaos, Solitons & Fractals, 2020, 130, 109405.

[11] R.M. Ganji, H. Jafari and S. Nemati, *A new approach for solving integro-differential equations of variable order*, Journal of Computational and Applied Mathematics, 2020, 379, 112946.

[12] M.H. Heydari, Z. Avazzadeh and Y. Yang, *A computational method for solving variable-order fractional nonlinear diffusion-wave equation*, Applied Mathematics and Computation, 2019, 352, 235–248.

[13] T. Zhou, M.N. Ncube, S. Salati and H. Jafari, *Operational matrices for solving a class of variable-order partial differential equations*, Fractals, 2025, Article, 2540130.

[14] H. Jafari, R.M. Ganji, S. Salati and S.J. Johnston, *A mixed-method to numerical simulation of variable order stochastic advection diffusion equations*, Alexandria Engineering Journal, 2024, 89, 60–70.

[15] S. Salati, H. Jafari and M. Matinfar, *A numerical method for solving a class of variable-order differential equations using Hosoya polynomial of simple paths*, Journal of Applied Mathematics and Computational Mechanics, 2024, 23(1), 97–108.

[16] M. Kashif, P. Pandey and H. Jafari, *A novel numerical manner for nonlinear coupled variable order reaction-diffusion equation*, Thermal Science, 2023, 27(Spec. issue 1), 353–363.

[17] R.P. Agarwal, V. Lakshmikantham and J.J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*,Nonlinear Analysis: Theory, Methods & Applications, 2010, 72(6), 2859–2862.

[18] S. Salahshour, T. Allahviranloo and S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Communications in Nonlinear Science and Numerical Simulation, 2012, 17(3), 1372–1381.

[19] T. Allahviranloo, S. Salahshour and S. Abbasbandy, *Explicit solutions of fractional differential equations with uncertainty*, Soft Computing, 2012, 16, 297–302.

[20] M. Mazandarani and A.V. Kamyad, *Modified fractional Euler method for solving fuzzy fractional initial value problem*, Communications in Nonlinear Science and Numerical Simulation, 2013, 18(1), 12–21.

[21] Z. Alijani, D. Baleanu, B. Shiri and G.C. Wu, *Spline collocation methods for systems of fuzzy fractional differential equations*, Chaos, Solitons & Fractals, 2020, 131, 109510.

[22] A. Ahmadian, S. Salahshour and C.S. Chan, *Fractional differential systems: a fuzzy solution based on operational matrix of shifted Chebyshev polynomials and its applications*, IEEE Transactions on Fuzzy Systems, 2016, 25(1), 218–236.

[23] A. Rivaz, O.S. Fard and T.A. Bidgoli, *Solving fuzzy fractional differential equations by a generalized differential transform method*, SeMA Journal, 2016, 73, 149–170.

[24] A. Ahmadian, F. Ismail, S. Salahshour, D. Baleanu and F. Ghaemi, *Uncertain viscoelastic models with fractional order: a new spectral tau method to study the numerical simulations of the solution*, Communications in Nonlinear Science and Numerical Simulation, 2017, 53, 44–64.

[25] B. Shiri, Z. Alijani and Y. Karaca, *A power series method for the fuzzy fractional logistic differential equation*, Fractals, 2023, 31(10), 2340086.

[26] Z. Alijani, B. Shiri, I. Perfilieva and D. Baleanu, *Numerical solution of a new mathematical model for intravenous drug administration*, Evolutionary Intelligence, 2024, 17(1), 559–575.

[27] H. Jafari, R.M. Ganji, D.D. Ganji, Z. Hammouch and Y.S. Gasimov, *A novel numerical method for solving fuzzy variable-order differential equations with Mittag-Leffler kernels*, Fractals, 2023, 31(4), 2340063.

[28] L.A. Zadeh, *Fuzzy sets*, Information and control, 1965, 8(3), 338–353.

[29] O. Kaleva, *A note on fuzzy differential equations*, Nonlinear Analysis: Theory, Methods & Applications, 2006, 64(5), 895–900.

[30] B. Shiri, D. Baleanu and C.Y. Ma, *Pathological study on uncertain numbers and proposed solutions for discrete fuzzy fractional order calculus*, Open Physics, 2023, 21(1), 20230135.

[31] B. Shiri, *A unified generalization for Hukuhara types differences and derivatives: Solid analysis and comparisons*, AIMS Math., 2023, 8(1), 2168–2190.

[32] B. Bede and S.G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets Syst., 2005, 151(3), 581–599.

[33] B. Bede and L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy sets and systems, 2013, 230, 119–141.

[34] T. Allahviranloo, *Fuzzy fractional differential operators and equations*, Studies in fuzziness and soft computing, 2021, 397.

[35] B. Shiri and D. Baleanu, *All linear fractional derivatives with power functions' convolution kernel and interpolation properties*, Chaos, Solitons & Fractals, 2023, 170, 113399.

[36] J.C. Mason and D.C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, 2002.