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Forced Epidemic Waves in a Nonlocal Dispersal SIR Model with Shifting Transmission and Time Delay*

Yilun Han[1], Yan Li[2] and Jiabing Wang[1,3,4,†]

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Abstract This paper is concerned with the forced epidemic waves of a nonlocal dispersal SIR model with shifting transmission and time delay. We first demonstrate that the existence of forced waves can be reduced to a fixed point problem. Then by constructing different pairs of upper and lower solutions and using Schauder's fixed point theorem, we establish two types of forced epidemic waves that reveal different state conversions of the disease. Moreover, we prove the nonexistence of forced epidemic waves when the basic reproduction number is less than unity. Finally, some biological explanations for the theoretical results are given in the discussion.

Keywords Forced waves, shifting transmission, nonlocal dispersal, SIR model, time delay

MSC(2010) 35K57, 35C07, 35R20, 92D30.

1. Introduction

Since the pioneering work of Kermack and McKendrick [21], compartmental models have played an important role in mathematical epidemiology, serving as powerful tools for analyzing and predicting disease spread. However, traditional compartmental models of ordinary differential equations (i.e., ODEs) cannot capture the spatial effects in disease transmission. Consequently, it is essential to consider reaction-diffusion systems in mathematical modeling. Compared to classical diffusion models, nonlocal dispersal models can accurately describe both the long-

[†]the corresponding author.
Email addresses: cughanyl@163.com (Y. Han), yanli@xidian.edu.cn (Y. Li), wangjb@cug.edu.cn (J. Wang)
[1]School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China.
[2]School of Mathematics and Statistics, Xidian University, Xi'an 710071, Shaanxi, China.
[3]Chongqing CUG Industrial Technology Research Institute, Chongqing 401336, China.
[4]Shenzhen Research Institute, China University of Geosciences, Shenzhen 518000, China.
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distance transmission of infectious diseases and the large-scale movement of host populations. Furthermore, for diseases with incubation periods, incorporating time delay into the model is a common approach. In particular, Li et al. [25] proposed the following nonlocal dispersal SIR model with time delay:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1[(J*S)(x,t) - S(x,t)] + B - \sigma S(x,t) - \dfrac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2[(J*I)(x,t) - I(x,t)] + \dfrac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma)I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3[(J*R)(x,t) - R(x,t)] + \gamma I(x,t) - \mu_1 R(x,t), \end{cases} \tag{1.1}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent the population sizes of susceptible, infective and removed class at location x and time t, respectively. $d_i > 0$ $(i = 1,2,3)$ are diffusion coefficients. $B > 0$ is the recruitment rate of the susceptible population. σ, μ and μ_1 are all positive constants and denote the death rates of each class. $\gamma > 0$ is the recovery rate while $\beta > 0$ is the infection rate. $\alpha > 0$ is the saturation parameter [8]. $\tau > 0$ is the time delay representing the incubation period of the disease. Furthermore, the nonlocal diffusion is characterized by the convolution operator $J * u(x,t) - u(x,t) := \int_{-\infty}^{+\infty} J(x-y)u(y,t)\mathrm{d}y - u(x,t)$, where J is the probability density function [3]. In recent years, the convolution operator above has been widely used to describe large-scale free movement of individuals in either ecological models [2,5,14,19,23,32,37] or epidemic models [13,20,25,26,34].

Since the third equation in (1.1) is decoupled from others, we only need to consider the first two equations. Introduce the following substitutions as

$$\widetilde{S}(x,t) = \frac{\sigma}{B} S\left(x, \frac{t}{d_2}\right), \ \widetilde{I}(x,t) = \frac{\sigma}{B} I\left(x, \frac{t}{d_2}\right),$$
$$\widetilde{\sigma} = \frac{\sigma}{d_2}, \ \widetilde{d} = \frac{d_1}{d_2}, \ \widetilde{\beta} = \frac{\beta B}{\sigma d_2}, \ \widetilde{\tau} = d_2\tau, \ \widetilde{\alpha} = \frac{\alpha B}{\sigma}, \ \widetilde{\mu} = \frac{\mu}{d_2}, \ \widetilde{\gamma} = \frac{\gamma}{d_2}$$

and drop the tilde for convenience, then the first two equations of (1.1) reduce to

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d[(J*S)(x,t) - S(x,t)] + \sigma - \sigma S(x,t) - \dfrac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = (J*I)(x,t) - I(x,t) + \dfrac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma)I(x,t). \end{cases} \tag{1.2}$$

For model (1.2), Li et al. [25] established the existence, nonexistence and minimal wave speed of traveling waves connecting the disease free equilibrium $(1,0)$ to the endemic equilibrium $\left(\frac{\mu+\gamma+\alpha\sigma}{\alpha\sigma+\beta}, \frac{\sigma[\beta-(\mu+\gamma)]}{(\mu+\gamma)(\alpha\sigma+\beta)}\right)$, and further illustrated how the latency of infection and the spatial movement of the infective individuals affect the minimal wave speed.

As we all know, climate change is one of the major challenges of the world today and has attracted global attention. In fact, climate change is an important factor in triggering diseases. For example, climate change affects the emergence of vector-borne diseases such as malaria, dengue and West Nile virus by altering their ranges, distribution or seasonality [27]. On the other hand, climate change also affects the transmission capacity of diseases. For instance, elevated temperatures can accelerate mosquito development, allowing them to mature faster and acquire the transmission capability of dengue fever and Zika virus [16]. Therefore, to investigate the impact of climate change on the dynamics of infectious diseases, it is reasonable to introduce the shifting effects of transmission rate into the infectious disease model. In this paper, we incorporate the shifting effects of transmission rate in (1.2) and consider the following epidemic system

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d[(J*S)(x,t) - S(x,t)] + \sigma - \sigma S(x,t) - \dfrac{\beta(x-ct)S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = (J*I)(x,t) - I(x,t) + \dfrac{\beta(x-ct)S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma)I(x,t), \end{cases} \tag{1.3}$$

where $\beta(x-ct)$ denotes the transmission rate function with $c>0$ being the shifting speed, which implies that the infectious capacity of the disease is related to the environment.

Forced waves, as a special class of traveling wave solutions of which the wave speed coincides with the shifting speed, have played an important role in describing the dynamics of the population in diffusive models. There is quite an extensive literature for the study of forced waves for the classical diffusion problems [4,6,7,9,10,15] or nonlocal diffusion problems [17, 22, 28–31, 35, 36]. However, to the best of our knowledge, there are only a few works that focus on forced waves in epidemic models [1, 11, 16, 33], which highlights the significance of studying such non-cooperative systems with shifting effects. The non-monotonicity and heterogeneity brought by the nonlocal term and shifting transmission cause some difficulties in studying model (1.3). To overcome these difficulties, we will use the method of upper and lower solutions combined with Schauder's fixed point theorem to obtain the existence of two types of forced waves. Furthermore, the nonexistence of forced waves is also proved by the comparison arguments. Because of the technical difficulties, the kernel function is assumed to be symmetrical and compactly supported. It is interesting and challenging to consider the case where the kernel function is asymmetric [24], we leave it for future consideration.

The rest of this paper is organized as follows. In Section 2, we present the basic assumptions and main results. In Section 3, we give some preliminaries. Section 4 is devoted to the existence of two types of forced waves. In Section 5, we prove the nonexistence of forced waves. Finally, some discussion is given in Section 6.

2. Main results

In this section, we state the basic assumptions and main results. Throughout the paper, the kernel function J always satisfies the following properties:

(J) $J \in C^1(\mathbb{R})$, $J(-x) = J(x) \geq 0$, $\int_{-\infty}^{\infty} J(x)dx = 1$ and J is compactly supported.

The shifting transmission rate $\beta(\cdot)$ satisfies:

(H1) $\beta(\xi) \in C(\mathbb{R})$ satisfies that $0 = \beta(-\infty) < \beta(\xi) \leq \beta_0 = \beta(+\infty)$ for all $\xi \in \mathbb{R}$ with β_0 being some positive constant;

(H2) $\beta_0 - \beta(\xi) \leq Ae^{-\theta\xi}$, $\forall \xi \geq K$ for a large positive number K and some positive constants A, θ;

(H3) $\beta(-\xi) \leq e^{-\tilde{\theta}\xi}$, $\forall \xi \geq \tilde{K}$ for a large positive number $\tilde{K}$ and some positive constant $\tilde{\theta}$.

In **(H1)**, the assumption $\beta(-\infty) = 0$ suggests that the disease will gradually lose its infectious capacity under unfavorable environment for disease transmission, while $\beta(+\infty) = \beta_0 > 0$ is the maximum infectious capacity of the disease.

When spatial diffusion is not considered, the following limiting delay differential equations (DDE) system of (1.3)

$$\begin{cases} \dfrac{dS}{dt} = \sigma - \sigma S(t) - \dfrac{\beta_0 S(t) I(t-\tau)}{1+\alpha I(t-\tau)}, \\ \dfrac{dI}{dt} = \dfrac{\beta_0 S(t) I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu+\gamma) I(t), \end{cases} \tag{2.1}$$

always admits a disease-free equilibrium $E_0 = (1, 0)$. By the theory of basic reproduction number $\mathcal{R}_0$ for general autonomous FDEs developed by Zhao [38], we have that $\mathcal{R}_0 := \frac{\beta_0}{\mu+\gamma}$ for model (2.1) regardless of whether the time delay is zero or not. When $\mathcal{R}_0 > 1$, system (1.3) admits a unique positive endemic equilibrium $E^* = (S^*, I^*)$, where

$$S^* = \frac{\mu+\gamma+\alpha\sigma}{\alpha\sigma+\beta_0}, \quad I^* = \frac{\sigma[\beta_0 - (\mu+\gamma)]}{(\mu+\gamma)(\alpha\sigma+\beta_0)}.$$

In this paper, we are mainly concerned with the existence of forced waves of (1.3). Now we put $(S, I)(x, t) = (\tilde{\phi}, \tilde{\psi})(\xi)$ with $\xi = x - ct$ and set $(\phi, \psi)(\xi) = (\tilde{\phi}, \tilde{\psi})(-\xi)$. By the symmetry of J, the wave equations corresponding to system (1.3) are

$$\begin{cases} c\phi'(\xi) = d\left[\displaystyle\int_{\mathbb{R}} J(y)\phi(\xi-y)\mathrm{d}y - \phi(\xi)\right] + \sigma - \sigma\phi(\xi) - \dfrac{\beta(-\xi)\phi(\xi)\psi(\xi-c\tau)}{1+\alpha\psi(\xi-c\tau)}, \\ c\psi'(\xi) = \displaystyle\int_{\mathbb{R}} J(y)\psi(\xi-y)\mathrm{d}y - \psi(\xi) + \dfrac{\beta(-\xi)\phi(\xi)\psi(\xi-c\tau)}{1+\alpha\psi(\xi-c\tau)} - (\mu+\gamma)\psi(\xi). \end{cases} \tag{2.2}$$

The main purpose of our research is to look for type-I wave connecting $(1, 0)$ and $(1, 0)$, and type-II wave connecting (S^*, I^*) and $(1, 0)$, which reads that

Type-I forced wave $\lim_{\xi\to-\infty} (\phi, \psi)(\xi) = (1, 0), \quad \lim_{\xi\to+\infty} (\phi, \psi)(\xi) = (1, 0);$

Type-II forced wave $\lim_{\xi\to-\infty} (\phi, \psi)(\xi) = (S^*, I^*), \quad \lim_{\xi\to+\infty} (\phi, \psi)(\xi) = (1, 0).$

Notice that for any nontrivial nonnegative bounded solution (ϕ, ψ) of (2.2), we have $0 < \phi < 1$ and $\psi > 0$ in $\mathbb{R}$. Indeed, if $\phi(\xi_0) = 0$, then $\phi'(\xi_0) = 0$, which is impossible by the first equation of (2.2). Similarly, any local maximal point ξ_0 of ϕ must have $\phi(\xi_0) \leq 1$. Further, by the strong maximum principles for nonlocal equations [12, Theorem 2.12], we obtain that $0 < \phi < 1$. The similar discussion shows that $\psi > 0$ in $\mathbb{R}$.

Our main results are summarized as follows.

Theorem 2.1. *(Existence of Type-I forced wave) Assume that **(H1)**, **(H2)** hold and $\mathcal{R}_0 > 1$. Then for any $c > c^*$ with c^* being defined in Lemma 4.1, there exists a nontrivial solution (ϕ, ψ) of (2.2) such that $(\phi, \psi)(-\infty) = (\phi, \psi)(+\infty) = (1, 0)$.*

Theorem 2.2. *(Existence of Type-II forced wave) Assume that **(H1)** and **(H3)** hold, and $\mathcal{R}_0 > 1$ and $\alpha > \alpha^* > \frac{\beta_0}{\sigma(\mathcal{R}_0 - 1)}$. Then for each $c > 0$, there exists a solution (ϕ, ψ) of* (2.2) *such that $(\phi, \psi)(-\infty) = (S^*, I^*)$ and $(\phi, \psi)(+\infty) = (1, 0)$.*

Theorem 2.3. *(Non-existence of forced waves) Suppose that $\mathcal{R}_0 < 1$. Then system* (2.2) *has no nontrivial nonnegative bounded solutions for any $c > 0$.*

3. Preliminaries

In this section, we will demonstrate that the existence of forced waves is equivalent to a fixed point problem. Then by using Schauder's fixed point theorem, the problem can be further reduced to constructing a pair of suitable upper and lower solutions. Furthermore, we consider the asymptotic behavior of these forced waves as time tends to positive infinity, which reveals the eventual extinction of the disease.

First, we introduce the following definition of upper and lower solutions of (2.2).

Definition 3.1. Continuous functions $(\phi_+(\xi), \psi_+(\xi))$ and $(\phi_-(\xi), \psi_-(\xi))$ are called a pair of upper and lower solutions of (2.2) if $\phi_- \le \phi_+, \ \ \psi_- \le \psi_+$ on $\mathbb{R}$ and satisfy the following inequalities

$$
\begin{aligned}
&d\left[\int_{\mathbb{R}} J(y)\phi_+(\xi-y)\mathrm{d}y - \phi_+\right] - c\phi_+' + \sigma - \sigma\phi_+ - \frac{\beta(-\xi)\phi_+\psi_-(\xi-c\tau)}{1+\alpha\psi_-(\xi-c\tau)} \le 0,\\
&\int_{\mathbb{R}} J(y)\psi_+(\xi-y)\mathrm{d}y - \psi_+ - c\psi_+' + \frac{\beta(-\xi)\phi_+\psi_+(\xi-c\tau)}{1+\alpha\psi_+(\xi-c\tau)} - (\mu+\gamma)\psi_+ \le 0,\\
&d\left[\int_{\mathbb{R}} J(y)\phi_-(\xi-y)\mathrm{d}y - \phi_-\right] - c\phi_-' + \sigma - \sigma\phi_- - \frac{\beta(-\xi)\phi_-\psi_+(\xi-c\tau)}{1+\alpha\psi_+(\xi-c\tau)} \ge 0,\\
&\int_{\mathbb{R}} J(y)\psi_-(\xi-y)\mathrm{d}y - \psi_- - c\psi_-' + \frac{\beta(-\xi)\phi_-\psi_-(\xi-c\tau)}{1+\alpha\psi_-(\xi-c\tau)} - (\mu+\gamma)\psi_- \ge 0,
\end{aligned}
$$

for $\xi \in \mathbb{R}\backslash E$ with some finite set E.

Remark 3.1. We can easily verify that $(0, 0)$ and $\left(1, \frac{\beta_0}{\alpha(\mu+\gamma)}\right)$ are a pair of constant upper and lower solutions for (2.2).

Denote

$$
\begin{aligned}
&P_1(\phi,\psi)(\xi)\\
&:= l\phi(\xi) + d\left[\int_{\mathbb{R}} J(y)\phi(\xi-y)\mathrm{d}y - \phi(\xi)\right] + \sigma - \sigma\phi(\xi) - \frac{\beta(-\xi)\phi(\xi)\psi(\xi-c\tau)}{1+\alpha\psi(\xi-c\tau)},\\
&P_2(\phi,\psi)(\xi)\\
&:= l\psi(\xi) + \int_{\mathbb{R}} J(y)\psi(\xi-y)\mathrm{d}y - \psi(\xi) + \frac{\beta(-\xi)\phi(\xi)\psi(\xi-c\tau)}{1+\alpha\psi(\xi-c\tau)} - (\mu+\gamma)\psi(\xi),
\end{aligned}
$$

where $l > 0$ is sufficiently large. Then system (2.2) can be re-written as

$$
\begin{cases}
c\phi'(\xi) = -l\phi(\xi) + P_1(\phi,\psi)(\xi),\\
c\psi'(\xi) = -l\psi(\xi) + P_2(\phi,\psi)(\xi).
\end{cases}
$$

Let $BUC(\mathbb{R})$ be the space of all bounded and uniformly continuous functions defined in $\mathbb{R}$ and denote $X := BUC(\mathbb{R}) \times BUC(\mathbb{R})$. Set

$$\omega := \left\{(\phi, \psi) \in X : 0 \le \phi \le 1, 0 \le \psi \le \frac{\beta_0}{\alpha(\mu+\gamma)}\right\}.$$

Define the following integral operator $H = (H_1, H_2) : \omega \to X$ by

$$H_1(\phi, \psi)(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{l}{c}(\xi - z)} P_1(\phi, \psi)(z)\mathrm{d}z,$$
$$H_2(\phi, \psi)(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{l}{c}(\xi - z)} P_2(\phi, \psi)(z)\mathrm{d}z.$$

In fact, $H(\phi, \psi)$ satisfies

$$\begin{cases} c[H_1(\phi, \psi)]'(\xi) + lH_1(\phi, \psi)(\xi) - P_1(\phi, \psi)(\xi) = 0, \\ c[H_2(\phi, \psi)]'(\xi) + lH_2(\phi, \psi)(\xi) - P_2(\phi, \psi)(\xi) = 0. \end{cases}$$

Thus, the existence of solutions of (2.2) reduces to find the fixed point of the operator $H(\phi, \psi)$.

For any given $\rho > 0$ and $(\phi, \psi) \in X$, define the norm

$$\|(\phi, \psi)\|_\rho := \sup_{\xi \in \mathbb{R}} |(\phi, \psi)(\xi)| e^{-\rho|\xi|},$$

where $|\cdot|$ is the usual norm on $\mathbb{R}^2$. It is easy to verify that $X_\rho := (X, \|\cdot\|_\rho)$ is a Banach space. We define

$$\Omega := \{(\phi, \psi) \in \omega : \phi_- \le \phi \le \phi_+, \ \ \psi_- \le \psi \le \psi_+ \ \text{ on } \mathbb{R}\},$$

and Ω is a non-empty, bounded, closed and convex subset of X_ρ.

Next, we show some properties of the operator H.

Lemma 3.1. *The operator H_1 is non-decreasing in ϕ and non-increasing in ψ while H_2 is non-decreasing in both ϕ and ψ. Further, H maps Ω into Ω.*

Proof. It is obvious that H_1 is non-increasing in ψ since P_1 is non-increasing in ψ. Now we prove that H_1 is non-decreasing in ϕ. Let $(\phi_1, \psi), (\phi_2, \psi) \in \Omega$ with $\phi_1 \ge \phi_2$. Then we have

$$\begin{aligned} &P_1(\phi_1, \psi)(\xi) - P_1(\phi_2, \psi)(\xi) \\ =&\left[l - d - \sigma - \frac{\beta(-\xi)\psi(\xi - c\tau)}{1 + \alpha\psi(\xi - c\tau)}\right](\phi_1(\xi) - \phi_2(\xi)) \\ &+ d\left[\int_{\mathbb{R}} J(y)(\phi_1(\xi - y) - \phi_2(\xi - y))\mathrm{d}y\right] \\ \ge&(l - d - \sigma - \frac{\beta_0}{\alpha})(\phi_1(\xi) - \phi_2(\xi)) + d\left[\int_{\mathbb{R}} J(y)(\phi_1(\xi - y) - \phi_2(\xi - y))\mathrm{d}y\right] \ge 0 \end{aligned}$$

since l is sufficiently large. This implies that

$$H_1(\phi_1, \psi)(\xi) - H_1(\phi_2, \psi)(\xi) \ge 0.$$

By a similar argument, it is easy to prove that H_2 is non-decreasing in both ϕ and ψ.

We now show that H maps Ω into Ω. For any $(\phi, \psi) \in \Omega$ and $\xi \in \mathbb{R}$, there holds that

$$H_1(\phi_-, \psi_+)(\xi) \le H_1(\phi, \psi)(\xi) \le H_1(\phi_+, \psi_-)(\xi),$$
$$H_2(\phi_-, \psi_-)(\xi) \le H_2(\phi, \psi)(\xi) \le H_2(\phi_+, \psi_+)(\xi).$$

Assume that $\xi_1 < \xi_2 < ... < \xi_n$. By Definition 3.1, we have

$$\begin{aligned}H_1(\phi_-, \psi_+)(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{l}{c}(\xi - z)} P_1(\phi_-, \psi_+)(z)\mathrm{d}z \\ &\ge \frac{1}{c}\left(\int_{-\infty}^{\xi_1} + \cdots + \int_{\xi_n}^{\xi}\right) e^{-\frac{l}{c}(\xi - z)}[c\phi_-'(z) + l\phi_-(z)]\mathrm{d}z \\ &= \phi_-(\xi).\end{aligned}$$

Similarly, we can prove that $H_1(\phi_+, \psi_-)(\xi) \le \phi_+(\xi), H_2(\phi_-, \psi_-)(\xi) \ge \psi_-(\xi)$ and $H_2(\phi_+, \psi_+)(\xi) \le \psi_+(\xi)$. Hence, H maps Ω into Ω. □

Lemma 3.2. *Given $c > 0$, suppose that (ϕ_+, ψ_+) and (ϕ_-, ψ_-) are a pair of upper and lower solutions of (2.2). Then (2.2) admits a solution (ϕ, ψ) with $\phi_- \le \phi \le \phi_+$ and $\psi_- \le \psi \le \psi_+$ on $\mathbb{R}$.*

Proof. We prove this lemma in three steps.

Step1: We first show that H is continuous with respect to the norm $\|\cdot\|_\rho$. By a direct calculation, we have

$$\begin{aligned}&|P_1(\phi_1, \psi_1)(\xi) - P_1(\phi_2, \psi_2)(\xi)|e^{-\rho|\xi|} \\ \le& l|\phi_1(\xi) - \phi_2(\xi)|e^{-\rho|\xi|} + d\int_{\mathbb{R}} J(y)|\phi_1(\xi - y) - \phi_2(\xi - y)|e^{-\rho|\xi|}\mathrm{d}y \\ &+ \left|\frac{\beta(-\xi)\phi_1(\xi)\psi_1(\xi - c\tau)}{1 + \alpha\psi_1(\xi - c\tau)} - \frac{\beta(-\xi)\phi_2(\xi)\psi_2(\xi - c\tau)}{1 + \alpha\psi_2(\xi - c\tau)}\right| e^{-\rho|\xi|} \\ :=& A_1 + A_2 + A_3,\end{aligned}$$

and

$$\begin{aligned}A_2 &:= d\int_{\mathbb{R}} J(y)|\phi_1(\xi - y) - \phi_2(\xi - y)|e^{-\rho|\xi|}\mathrm{d}y \\ &= d\int_{\mathbb{R}} J(y)|\phi_1(\xi - y) - \phi_2(\xi - y)|e^{-\rho|\xi|}e^{-\rho|y|}e^{\rho|y|}\mathrm{d}y \\ &\le d\int_{\mathbb{R}} J(y)|\phi_1(\xi - y) - \phi_2(\xi - y)|e^{-\rho|\xi - y|}e^{\rho|y|}\mathrm{d}y \\ &\le d\int_{-L}^{L} J(y)e^{\rho|y|}\mathrm{d}y\|(\phi_1 - \phi_2, \psi_1 - \psi_2)\|_\rho \\ &= \mathcal{J}_\rho\|(\phi_1 - \phi_2, \psi_1 - \psi_2)\|_\rho,\end{aligned}$$

where $L > 0$ is the radius of $SuppJ$ and $\mathcal{J}_\rho := d\int_{-L}^{L} J(y)e^{\rho|y|}\mathrm{d}y$.

And we have

$$\begin{aligned}
A_3 :=& \left|\frac{\beta(-\xi)\phi_1(\xi)\psi_1(\xi-c\tau)}{1+\alpha\psi_1(\xi-c\tau)} - \frac{\beta(-\xi)\phi_2(\xi)\psi_2(\xi-c\tau)}{1+\alpha\psi_2(\xi-c\tau)}\right| e^{-\rho|\xi|}\\
\le& \frac{\beta(-\xi)\psi_1(\xi-c\tau)}{1+\alpha\psi_1(\xi-c\tau)}|\phi_1(\xi)-\phi_2(\xi)|e^{-\rho|\xi|}\\
&+\frac{\beta(-\xi)\phi_2(\xi)|\psi_1(\xi-c\tau)-\psi_2(\xi-c\tau)|e^{-\rho|\xi|}}{(1+\alpha\psi_1(\xi-c\tau))(1+\alpha\psi_2(\xi-c\tau))}\\
\le& \frac{\beta_0}{\alpha}|\phi_1(\xi)-\phi_2(\xi)|e^{-\rho|\xi|} + \beta_0 e^{\rho c\tau}|\psi_1(\xi-c\tau)-\psi_2(\xi-c\tau)|e^{-\rho|\xi-c\tau|}\\
\le& (\frac{\beta_0}{\alpha}+\beta_0 e^{\rho c\tau})\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_\rho.
\end{aligned}$$

Thus we have

$$|P_1(\phi_1,\psi_1)(\xi)-P_1(\phi_2,\psi_2)(\xi)|e^{-\rho|\xi|} \le B_1\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_\rho,$$

where $B_1 := l+\mathcal{J}_\rho+\frac{\beta_0}{\alpha}+\beta_0 e^{\rho c\tau}$. Similarly, we have

$$|P_2(\phi_1,\psi_1)(\xi)-P_2(\phi_2,\psi_2)(\xi)|e^{-\rho|\xi|} \le B_2\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_\rho$$

for a positive constant B_2 and denote $B := \max\{B_1,B_2\}$. Thus, for $k=1,2$,

$$\begin{aligned}
&|H_k(\phi_1,\psi_1)(\xi)-H_k(\phi_2,\psi_2)(\xi)|\\
\le& \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{l}{c}(\xi-z)}\left|P_k(\phi_1,\psi_1)(z)-P_k(\phi_2,\psi_2)(z)\right| e^{-\rho|z|}e^{\rho|z|}\mathrm{d}z\\
\le& \frac{B}{c}\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_\rho\int_{-\infty}^{\xi} e^{-\frac{l}{c}(\xi-z)}e^{\rho|z|}\mathrm{d}z.
\end{aligned}$$

For $\xi\ge 0$, by a direct calculation, it follows that

$$|H_k(\phi_1,\psi_1)(\xi)-H_k(\phi_2,\psi_2)(\xi)|e^{-\rho|\xi|} \le B(\frac{1}{l-c\rho}+\frac{1}{l+c\rho})\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_\rho.$$

For $\xi<0$, it follows that

$$|H_k(\phi_1,\psi_1)(\xi)-H_k(\phi_2,\psi_2)(\xi)|e^{-\rho|\xi|} \le \frac{B}{l-c\rho}\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_\rho.$$

Step2: We now show that $H:\Omega\to\Omega$ is a compact operator. By a direct calculation, we have

$$\left|\frac{\mathrm{d}H_1(\phi,\psi)(\xi)}{\mathrm{d}\xi}\right| \le \frac{l+d+\sigma}{c}\left[1+\int_{-\infty}^{\xi}\frac{l}{c}e^{-\frac{l}{c}(\xi-z)}\mathrm{d}z\right] = \frac{2(l+d+\sigma)}{c}.$$

Similarly,

$$\left|\frac{\mathrm{d}H_2(\phi,\psi)(\xi)}{\mathrm{d}\xi}\right| \le \frac{2}{c}\left[\frac{\beta_0(l+1)}{\alpha(\mu+\gamma)}+\frac{\beta_0}{\alpha}\right].$$

Hence, $H(\Omega)$ is equi-continuous. For $(\phi_n,\psi_n)\in\Omega$, denote $\tilde\phi_n = H_1(\phi_n,\psi_n), \tilde\psi_n = H_2(\phi_n,\psi_n)$. Since $\{(\tilde\phi_n,\tilde\psi_n)\}$ is uniformly bounded and equi-continuous on $[-T,T]$

for any sufficiently large number T. By Arzela-Ascoli's theorem, $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$ has a subsequence, which is convergent on $[-T, T]$ with respect to the supremum norm. We still denote the subsequence by $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$ for convenience. Then $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$ is a Cauchy sequence on $[-T, T]$ with respect to the supremum norm.

Given $\epsilon > 0$, choose $T > 0$ large enough such that

$$\sup_{|\xi|\geq T} \sqrt{[\tilde{\phi}_k(\xi) - \tilde{\phi}_l(\xi)]^2 + [\tilde{\psi}_k(\xi) - \tilde{\psi}_l(\xi)]^2} \cdot e^{-\rho|\xi|} \leq 2\sqrt{1^2 + \left[\frac{\beta_0}{\alpha(\mu+\gamma)}\right]^2} \cdot e^{-\rho T} < \frac{\epsilon}{2}.$$

Since $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$ is a Cauchy sequence on $[-T, T]$, there exist $N > 0$ and $k, l > N$ such that

$$\begin{aligned}
&\sup_{|\xi|\leq T} \sqrt{[\tilde{\phi}_k(\xi) - \tilde{\phi}_l(\xi)]^2 + [\tilde{\psi}_k(\xi) - \tilde{\psi}_l(\xi)]^2} \cdot \mathrm{e}^{-\rho|\xi|} \\
\leq &\sup_{|\xi|\leq T} \sqrt{[\tilde{\phi}_k(\xi) - \tilde{\phi}_l(\xi)]^2 + [\tilde{\psi}_k(\xi) - \tilde{\psi}_l(\xi)]^2} < \frac{\epsilon}{2}.
\end{aligned}$$

We obtain that $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$ is a Cauchy sequence. Since X_ρ is a Banach space, $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$ is convergent with respect to $\|\cdot\|_\rho$.

We further define the sequence of operators $\{H^T\}_{T=1}^{\infty}$ as

$$H^T(\phi, \psi)(\xi) := \begin{cases} H(\phi, \psi)(-T), & \xi < -T, \\ H(\phi, \psi)(\xi), & \xi \in [-T, T], \\ H(\phi, \psi)(T), & \xi > T. \end{cases}$$

By the above argument, we know that $H^T(\phi, \psi)(\xi)$ is compact. Moreover

$$\left|H^T(\phi, \psi)(\xi) - H(\phi, \psi)(\xi)\right| e^{-\rho|\xi|} \leq Me^{-\rho T}$$

for some positive constant M. Then $H^T(\phi, \psi)(\xi)$ uniformly converges to $H(\phi, \psi)(\xi)$ with respect to T. Consequently, $H(\phi, \psi)(\xi)$ is compact.

Step3: By using Schauder's fixed point theorem on the bounded, closed and convex set Ω, we obtain that H admits a fixed point in Ω. We complete the proof. □

At the end of this section, we give a universal lemma to illustrate the right-hand tail limit.

Lemma 3.3. *Assume that **(H1)** holds. Let (ϕ, ψ) be a nonnegative bounded solution of (2.2) for any given $c > 0$. Then $(\phi, \psi)(+\infty) = (1, 0)$.*

Proof. We first prove that $\psi(+\infty) = 0$.

On the contrary, we assume that $\bar{\psi} := \limsup_{\xi\to+\infty} \psi(\xi) > 0$. If ψ is oscillatory near $\xi = +\infty$, then there exists a maximal sequence $\{\xi_n\}$ such that $\lim_{n\to+\infty} \xi_n = +\infty$, $\lim_{n\to+\infty} \psi(\xi_n) = \bar{\psi}$ and $\psi'(\xi_n) = 0$. By Fatou's lemma, we have

$$\limsup_{n\to+\infty} \left\{\int_{\mathbb{R}} J(y)\psi(\xi_n - y)\mathrm{d}y\right\} \leq \int_{\mathbb{R}} \limsup_{n\to+\infty} J(y)\psi(\xi_n - y)\mathrm{d}y = \bar{\psi}.$$

Substituting $\{\xi_n\}$ in the ψ-equation of (2.2) and taking the upper limit on both sides, we obtain that

$$\begin{aligned}0 &= \limsup_{n\to+\infty}\left\{\int_{\mathbb{R}} J(y)\psi(\xi_n - y)\mathrm{d}y - \psi(\xi_n) + \frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)} - (\mu+\gamma)\psi(\xi_n)\right\}\\ &\leq \limsup_{n\to+\infty}\left\{\int_{\mathbb{R}} J(y)\psi(\xi_n - y)\mathrm{d}y\right\} + \limsup_{n\to+\infty}\{-\psi(\xi_n)\}\\ &\quad + \limsup_{n\to+\infty}\left\{\frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)}\right\} + \limsup_{n\to+\infty}\{-(\mu+\gamma)\psi(\xi_n)\}\\ &= \limsup_{n\to+\infty}\left\{\int_{\mathbb{R}} J(y)\psi(\xi_n - y)\mathrm{d}y\right\} - \bar{\psi} - (\mu+\gamma)\bar{\psi} < 0.\end{aligned}$$

This is a contradiction. On the other hand, if $\psi(\xi)$ is monotone near $\xi = +\infty$, then $\psi(+\infty) = \bar{\psi}$.

For any $n \in \mathbb{R}$, we have

$$\begin{aligned}\int_0^n \int_{\mathbb{R}} J(y)\left[\psi(\xi) - \psi(\xi - y)\right]\mathrm{d}y\mathrm{d}\xi &= \int_0^n \int_{\mathbb{R}} J(y)y\int_0^1 \psi'(\xi - sy)\mathrm{d}s\mathrm{d}y\mathrm{d}\xi\\ &= \int_{\mathbb{R}} J(y)y\int_0^1\int_0^n \psi'(\xi - sy)\mathrm{d}\xi\mathrm{d}s\mathrm{d}y\\ &= \int_{\mathbb{R}} J(y)y\int_0^1 \left[\psi(n-sy) - \psi(-sy)\right]\mathrm{d}s\mathrm{d}y.\end{aligned}$$

Integrating the ψ-equation of (2.2) from 0 to n, we have

$$\begin{aligned}c\left[\psi(n) - \psi(0)\right] &+ \int_{\mathbb{R}} J(y)y\int_0^1 \left[\psi(n-sy) - \psi(-sy)\right]\mathrm{d}s\mathrm{d}y\\ &= \int_0^n \frac{\beta(-\xi)\phi(\xi)\psi(\xi - c\tau)}{1+\alpha\psi(\xi - c\tau)} - (\mu+\gamma)\psi(\xi)\mathrm{d}\xi.\end{aligned} \tag{3.1}$$

Note that the left-hand part of (3.1) is uniformly bounded with respect to n. Since $\beta(-\infty) = 0, \psi(+\infty) = \bar{\psi}$ and both ϕ and ψ are bounded, we can find a positive number K such that

$$\frac{\beta(-\xi)\phi(\xi)\psi(\xi - c\tau)}{1+\alpha\psi(\xi - c\tau)} \leq (\mu+\gamma)\frac{\bar{\psi}}{4},\quad \psi(\xi) \geq \frac{\bar{\psi}}{2}\ \text{ for }\ \xi \geq K.$$

Thus

$$\frac{\beta(-\xi)\phi(\xi)\psi(\xi - c\tau)}{1+\alpha\psi(\xi - c\tau)} - (\mu+\gamma)\psi(\xi) \leq -\frac{\bar{\psi}}{4}(\mu+\gamma)\ \text{ for }\ \xi \geq K.$$

When n tends to $+\infty$, the right-hand part of (3.1) is unbounded. However, the left-hand of (3.1) is uniformly bounded with respect to n. We derive a contradiction and hence $\psi(+\infty) = 0$.

To show $\phi(+\infty) = 1$, on the contrary, suppose that $\liminf_{\xi\to+\infty}\phi(\xi) < 1$. In view of the ϕ-equation of (2.2) and $\beta(-\infty) = \psi(+\infty) = 0$, by a similar argument, we can easily show that $\phi(+\infty) = 1$. Hence, we complete the proof. □

4. Existence of forced waves

In this section, we are devoted to the existence of forced waves of (1.3).

4.1. Existence of Type-I wave

For $\lambda > 0$ and $c > 0$, define

$$\Delta(\lambda, c) = \int_{\mathbb{R}} J(y)e^{-\lambda y}\mathrm{d}y - c\lambda + \beta_0 e^{-\lambda c\tau} - (\mu + \gamma + 1).$$

By a simple calculation, we have

$$\begin{aligned}
&\Delta(0, c) = \beta_0 - (\mu + \gamma), \ \ \forall c > 0, \ \ \Delta(\lambda, +\infty) = -\infty, \ \ \forall \lambda > 0,\\
&\frac{\partial \Delta(\lambda, c)}{\partial c} = -\lambda - \lambda\tau\beta_0 e^{-\lambda c\tau} < 0, \ \ \frac{\partial \Delta(0, c)}{\partial \lambda} = -c - c\tau\beta_0 < 0,\\
&\frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} = \int_{\mathbb{R}} J(y)y^2 e^{-\lambda y}\mathrm{d}y + c^2\tau^2\beta_0 e^{-\lambda c\tau} > 0.
\end{aligned}$$

These basic properties of $\Delta(\lambda, c)$ imply the following lemma.

Lemma 4.1. *Suppose that $\mathcal{R}_0 := \frac{\beta_0}{\mu+\gamma} > 1$. Then there exists $c^* > 0$ and $\lambda^* > 0$ such that*

$$\Delta(\lambda^*, c^*) = 0, \quad \frac{\partial \Delta(\lambda, c)}{\partial \lambda}|_{(\lambda^*, c^*)} = 0.$$

Furthermore,

(i) $\Delta(\lambda, c) > 0$ for any $\lambda > 0$ and $0 < c < c^*$.

(ii) If $c > c^*$, then $\Delta(\lambda, c) = 0$ has two different positive solutions $\lambda_1 < \lambda_2$ satisfying

$$\Delta(\lambda, c) \begin{cases} > 0, & \lambda \in (0, \lambda_1) \cup (\lambda_2, \infty), \\ < 0, & \lambda \in (\lambda_1, \lambda_2). \end{cases}$$

Lemma 4.2. *Given $c > c^*$. Assume that $\mathcal{R}_0 > 1$ and **(H1)**, **(H2)** hold. Define continuous functions as follows*

$$\begin{aligned}
&\phi_+(\xi) \equiv 1, && \psi_+(\xi) = \min\left\{e^{\lambda_1\xi}, \tfrac{\beta_0}{\alpha(\mu+\gamma)}\right\},\\
&\phi_-(\xi) = \max\{1 - L_0 e^{\eta_0\xi}, 0\}, && \psi_-(\xi) = \max\left\{e^{\lambda_1\xi}(1 - L_1 e^{\eta_1\xi}), 0\right\},
\end{aligned}$$

where η_0, η_1, L_0, L_1 are positive constants to be determined later. Then (ϕ_+, ψ_+) and (ϕ_-, ψ_-) are a pair of upper and lower solutions for (2.2).

Proof. The first inequality in Definition 3.1 holds naturally. In view of the definition of $\psi_+(\xi)$, we have

$$\int_{\mathbb{R}} J(y)\psi_+(\xi - y)\mathrm{d}y \le \min\left\{e^{\lambda_1\xi}\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}\mathrm{d}y, \ \frac{\beta_0}{\alpha(\mu+\gamma)}\right\}.$$

Let $\xi_1 := \frac{1}{\lambda_1}\ln\frac{\beta_0}{\alpha(\mu+\gamma)}$. If $\xi > \xi_1$, then $\psi_+(\xi) = \frac{\beta_0}{\alpha(\mu+\gamma)}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} J(y)\psi_+(\xi - y)\mathrm{d}y - \psi_+(\xi) - c\psi_+'(\xi) + \frac{\beta(-\xi)\phi_+(\xi)\psi_+(\xi - c\tau)}{1 + \alpha\psi_+(\xi - c\tau)} - (\mu + \gamma)\psi_+(\xi)\\
\le&\frac{\beta_0}{\alpha} - \frac{\beta_0}{\alpha} = 0.
\end{aligned}$$

If $\xi < \xi_1$, then $\psi_+(\xi) = e^{\lambda_1 \xi}$. We have

$$\begin{aligned}&\int_{\mathbb{R}} J(y)\psi_+(\xi - y)\mathrm{d}y - \psi_+(\xi) - c\psi_+'(\xi) + \frac{\beta(-\xi)\phi_+(\xi)\psi_+(\xi - c\tau)}{1 + \alpha\psi_+(\xi - c\tau)} - (\mu + \gamma)\psi_+(\xi)\\ \le & e^{\lambda_1\xi}\left[\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}\mathrm{d}y - c\lambda_1 - (\mu + \gamma + 1) + \beta_0 e^{-\lambda_1 c\tau}\right] = 0.\end{aligned}$$

Thus the second inequality in Definition 3.1 holds.

In view of the definition of $\phi_-(\xi)$, we have

$$\int_{\mathbb{R}} J(y)\phi_-(\xi - y)\mathrm{d}y \ge \max\left\{1 - L_0 e^{\eta_0\xi}\int_{\mathbb{R}} J(y)e^{-\eta_0 y}\mathrm{d}y,\ 0\right\}.$$

Choose $\eta_0 > 0$ and $L_0 > 1$. Then $\xi_2 := \frac{1}{\eta_0}\ln\frac{1}{L_0} < 0$. If $\xi > \xi_2$, then $\phi_-(\xi) = 0$ and the third inequality of Definition 3.1 holds naturally. If $\xi < \xi_2 < 0$, then $\phi_-(\xi) = 1 - L_0 e^{\eta_0\xi}$. By calculations, we have

$$\begin{aligned}& d\left[\int_{\mathbb{R}} J(y)\phi_-(\xi - y)\mathrm{d}y - \phi_-(\xi)\right] - c\phi_-'(\xi) + \sigma - \sigma\phi_-(\xi) - \frac{\beta(-\xi)\phi_-(\xi)\psi_+(\xi - c\tau)}{1 + \alpha\psi_+(\xi - c\tau)}\\ \ge & L_0 e^{\eta_0\xi}\left\{-d\int_{\mathbb{R}} J(y)[e^{-\eta_0 y} - 1]\mathrm{d}y + c\eta_0 + \sigma\right\} - \frac{\beta_0(1 - L_0 e^{\eta_0\xi})e^{\lambda_1(\xi - c\tau)}}{1 + \alpha e^{\lambda_1(\xi - c\tau)}}\\ \ge & L_0 e^{\eta_0\xi}\left\{-d\int_{\mathbb{R}} J(y)[e^{-\eta_0 y} - 1]\mathrm{d}y + c\eta_0 + \sigma\right\} - \beta_0 e^{\lambda_1\xi}.\end{aligned}$$

By continuity, we can take $\eta_0 \in (0, \lambda_1)$ small enough such that

$$-d\int_{\mathbb{R}} J(y)[e^{-\eta_0 y} - 1]\mathrm{d}y + c\eta_0 + \sigma > 0.$$

Note that $e^{(\lambda_1 - \eta_0)\xi} < 1$. Hence, if we take

$$L_0 = \frac{\beta_0}{-d\int_{\mathbb{R}} J(y)[e^{-\eta_0 y} - 1]\mathrm{d}y + c\eta_0 + \sigma} + 1,$$

then the third inequality of Definition 3.1 holds.

In view of the definition of $\psi_-(\xi)$, we have

$$\begin{aligned}&\int_{\mathbb{R}} J(y)\psi_-(\xi - y)\mathrm{d}y\\ &\ge \max\left\{e^{\lambda_1\xi}\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}\mathrm{d}y - L_1 e^{(\lambda_1 + \eta_1)\xi}\int_{\mathbb{R}} J(y)e^{-(\lambda_1 + \eta_1)y}\mathrm{d}y, 0\right\}.\end{aligned}$$

Choose $\eta_1 = \frac{1}{2}\min\{\eta_0, \lambda_1, \lambda_2 - \lambda_1, \theta\}$ (see θ in assumption **(H2)**). Denote $\xi_3 := \frac{1}{\eta_1}\ln\frac{1}{L_1}$ and let $L_1 > L^*$ be sufficiently large such that $\xi_3 < \xi_2 < 0$ and $\xi_3 < -K$. If $\xi > \xi_3$, then $\psi_- = 0$ and the fourth inequality of Definition 3.1 holds naturally. If $\xi < \xi_3$, then $\phi_-(\xi) = 1 - L_0 e^{\eta_0\xi}$ and $\psi_-(\xi) = e^{\lambda_1\xi}(1 - L_1 e^{\eta_1\xi})$. In view of assumption **(H2)** and by a direct calculation, we have

$$\begin{aligned}
\frac{\beta(-\xi)\phi_-(\xi)\psi_-(\xi-c\tau)}{1+\alpha\psi_-(\xi-c\tau)} =&\frac{\beta(-\xi)\phi_-(\xi)\left[e^{\lambda_1(\xi-c\tau)}-L_1e^{(\lambda_1+\eta_1)(\xi-c\tau)}\right]}{1+\alpha\psi_-(\xi-c\tau)}\\
=&\frac{\beta(-\xi)\phi_-(\xi)e^{\lambda_1(\xi-c\tau)}}{1+\alpha\psi_-(\xi-c\tau)}-\frac{\beta(-\xi)\phi_-(\xi)L_1e^{(\lambda_1+\eta_1)(\xi-c\tau)}}{1+\alpha\psi_-(\xi-c\tau)}\\
\geq&\frac{\beta(-\xi)(1-L_0e^{\eta_0\xi})e^{\lambda_1(\xi-c\tau)}}{1+\alpha\psi_-(\xi-c\tau)}-\beta_0L_1e^{(\lambda_1+\eta_1)(\xi-c\tau)}\\
\geq&\frac{\beta(-\xi)e^{\lambda_1(\xi-c\tau)}}{1+\alpha\psi_-(\xi-c\tau)}-\beta_0L_0e^{(\eta_0+\lambda_1)\xi}-\beta_0L_1e^{(\lambda_1+\eta_1)(\xi-c\tau)}\\
=&\left[\frac{\beta(-\xi)e^{\lambda_1(\xi-c\tau)}}{1+\alpha\psi_-(\xi-c\tau)}-\beta_0e^{\lambda_1(\xi-c\tau)}\right]+\beta_0e^{\lambda_1(\xi-c\tau)}\\
&-\beta_0L_0e^{(\eta_0+\lambda_1)\xi}-\beta_0L_1e^{(\lambda_1+\eta_1)(\xi-c\tau)}\\
\geq&-Ae^{(\lambda_1+\theta)\xi}-\alpha\beta_0e^{2\lambda_1\xi}+\beta_0e^{\lambda_1(\xi-c\tau)}\\
&-\beta_0L_0e^{(\eta_0+\lambda_1)\xi}-\beta_0L_1e^{(\lambda_1+\eta_1)(\xi-c\tau)}\\
:=&\Lambda(\xi).
\end{aligned}$$

Then it suffices to show that

$$\int_{\mathbb{R}}J(y)\psi_-(\xi-y)\mathrm{d}y-\psi_-(\xi)-c\psi_-'(\xi)+\Lambda(\xi)-(\mu+\gamma)\psi_-(\xi)\geq 0,$$

which is equivalent to the following inequality

$$e^{\lambda_1\xi}\Delta(\lambda_1,c)-L_1e^{(\lambda_1+\eta_1)\xi}\Delta(\lambda_1+\eta_1,c)-Ae^{(\lambda_1+\theta)\xi}-\alpha\beta_0e^{2\lambda_1\xi}-\beta_0L_0e^{(\eta_0+\lambda_1)\xi}\geq 0.$$

In view of Lemma 4.1, we know that

$$\Delta(\lambda_1,c)=0,\ \ \Delta(\lambda_1+\eta_1,c)<0.$$

Then it suffices to show

$$L_1\geq\frac{Ae^{(\theta-\eta_1)\xi}+\alpha\beta_0e^{(\lambda_1-\eta_1)\xi}+\beta_0L_0e^{(\eta_0-\eta_1)\xi}}{-\Delta(\lambda_1+\eta_1,c)}.$$

By the choice of η_1, we know that

$$e^{(\theta-\eta_1)\xi}<1,\ \ e^{(\lambda_1-\eta_1)\xi}<1,\ \ e^{(\eta_0-\eta_1)\xi}<1.$$

Taking

$$L_1=\max\left\{\frac{A+\alpha\beta_0+L_0\beta_0}{-\Delta(\lambda_1+\eta_1,c)}+1,\ \ L^*\right\},$$

then the fourth inequality of Definition 3.1 holds. Here we complete the proof of the lemma. □

Proof. [**Proof of Theorem 2.1.**] In view of Lemmas 3.2, 3.3 and 4.2, we see that for each $c>c^*$, there exists a nonnegative solution (ϕ,ψ) of (2.2) satisfying $(\phi,\psi)(-\infty)=(\phi,\psi)(+\infty)=(1,0)$. Further, we show that the solution obtained above is indeed nontrivial. From the construction of the lower solutions in Lemma 4.2, we know that $\psi_-(\xi)\not\equiv 0$, and hence $\psi(\xi)\not\equiv 0$. Assume on the contrary that $\phi(\xi)\equiv 1$. Then in view of the first equation in (2.2) and the assumption $\beta(\cdot)>0$ in **(H1)**, we have that $\psi(\xi)\equiv 0$. This is a contradiction, which indicates the solution is nontrivial. □

4.2. Existence of Type-II wave

In order to show the existence of Type-II wave, we need to construct another pair of upper and lower solutions. For convenience, we denote

$$\phi^* = \frac{\sigma}{\sigma + \frac{\beta_0}{\alpha}}, \quad \psi^* = \frac{\beta_0 - (\mu+\gamma)}{\alpha(\mu+\gamma)} = \frac{1}{\alpha}(\mathcal{R}_0 - 1).$$

Lemma 4.3. *Assume that $R_0 > 1$ and **(H1)**, **(H3)** hold. Let $\alpha > \frac{\beta_0}{\sigma(\mathcal{R}_0-1)}$ and $c > 0$. Define continuous functions as follows*

$$\begin{aligned} &\phi_+(\xi) \equiv 1, && \psi_+(\xi) \equiv \psi^*, \\ &\phi_-(\xi) = \max\{1 - M_0 e^{-\nu_0 \xi}, \phi^*\}, && \psi_-(\xi) = \max\left\{\delta(1 - M_1 e^{\nu_1 \xi}), 0\right\}, \end{aligned}$$

where $\nu_0, \nu_1, \delta, M_0, M_1$ are positive constants to be determined later. Then (ϕ_+, ψ_+) and (ϕ_-, ψ_-) are a pair of upper and lower solutions for (2.2).

Proof. The first inequality in Definition 3.1 holds naturally. Since $\psi_+(\xi) \equiv \psi^*$, then

$$\begin{aligned} &\int_{\mathbb{R}} J(y)\psi_+(\xi - y)\mathrm{d}y - \psi_+(\xi) - c\psi_+'(\xi) + \frac{\beta(-\xi)\phi_+(\xi)\psi_+(\xi - c\tau)}{1 + \alpha\psi_+(\xi - c\tau)} - (\mu+\gamma)\psi_+(\xi) \\ \le& \frac{\beta_0 \psi^*}{1 + \alpha\psi^*} - (\mu+\gamma)\psi^* = 0. \end{aligned}$$

Thus the second inequality in Definition 3.1 holds.

By continuity, we can find a small $\nu_0 \in (0, \tilde{\theta})$ (see $\tilde{\theta}$ in assumption **(H3)**) such that

$$-d \int_{\mathbb{R}} J(y)\left[e^{\nu_0 y} - 1\right] \mathrm{d}y - c\nu_0 + \sigma > 0.$$

For $\phi_-(\xi)$, there exists $\zeta_1 \in \mathbb{R}$ such that $\phi_-(\xi) = \phi^*$ for $\xi < \zeta_1$ and $\phi_-(\xi) = 1 - M_0 e^{-\nu_0 \xi}$ for $\xi > \zeta_1$. Choose M_0 large enough such that $\zeta_1 > \tilde{K} > 0$ and

$$M_0 \ge \frac{1}{\alpha\left\{-d\int_{\mathbb{R}} J(y)\left[e^{\nu_0 y} - 1\right]\mathrm{d}y - c\nu_0 + \sigma\right\}}.$$

In view of the definition of $\phi_-(\xi)$, we have

$$\int_{\mathbb{R}} J(y)\phi_-(\xi - y)\mathrm{d}y \ge \max\left\{1 - M_0 e^{-\nu_0 \xi}\int_{\mathbb{R}} J(y) e^{\nu_0 y}\mathrm{d}y,\ \ \phi^*\right\}.$$

If $\xi < \zeta_1$, then $\phi_-(\xi) = \phi^*$. We have

$$\begin{aligned} &d\left[\int_{\mathbb{R}} J(y)\phi_-(\xi - y)\mathrm{d}y - \phi_-(\xi)\right] - c\phi_-'(\xi) + \sigma - \sigma\phi_-(\xi) - \frac{\beta(-\xi)\phi_-(\xi)\psi_+(\xi - c\tau)}{1 + \alpha\psi_+(\xi - c\tau)} \\ \ge& \sigma - \sigma\phi^* - \frac{\beta_0}{\alpha}\phi^* = 0. \end{aligned}$$

If $\xi > \zeta_1 > \tilde{K}$, then $\phi_-(\xi) = 1 - M_0 e^{-\nu_0 \xi}$. In view of assumption **(H3)**, we have

$$\begin{aligned} &d\left[\int_{\mathbb{R}} J(y)\phi_-(\xi - y)\mathrm{d}y - \phi_-(\xi)\right] - c\phi_-'(\xi) + \sigma - \sigma\phi_-(\xi) - \frac{\beta(-\xi)\phi_-(\xi)\psi_+(\xi - c\tau)}{1 + \alpha\psi_+(\xi - c\tau)} \\ \ge& M_0 e^{-\nu_0 \xi}\left\{-d\int_{\mathbb{R}} J(y)\left[e^{\nu_0 y} - 1\right]\mathrm{d}y - c\nu_0 + \sigma\right\} - \frac{1}{\alpha}e^{-\tilde{\theta}\xi} \\ \ge& e^{-\nu_0 \xi}\left\{M_0\left[-d\int_{\mathbb{R}} J(y)\left[e^{\nu_0 y} - 1\right]\mathrm{d}y - c\nu_0 + \sigma\right] - \frac{1}{\alpha}\right\} \ge 0. \end{aligned}$$

Hence the third inequality of Definition 3.1 holds.

Now we prove the last inequality of Definition 3.1. For a fixed $c > 0$, define the following function

$$F(\lambda) = -\int_{\mathbb{R}} J(y)\left[e^{-\lambda y} - 1\right] \mathrm{d}y + c\lambda.$$

By a simple calculation, we have $F(0) = 0$ and $F'(0) = c > 0$. Then we can fix a small $\nu_1 > 0$ such that

$$F(\nu_1) = -\int_{\mathbb{R}} J(y)\left[e^{-\nu_1 y} - 1\right] \mathrm{d}y + c\nu_1 > 0.$$

Note that $\alpha > \frac{\beta_0}{\sigma(\mathcal{R}_0 - 1)}$ is equivalent to $\beta_0\phi^* - (\mu+\gamma) > 0$. Thus we can choose some small positive constants ϵ and δ such that

$$\frac{(\beta_0 - \epsilon)\phi^*}{1+\alpha\delta} - (\mu+\gamma) > 0.$$

Let $\zeta_2 := \frac{1}{\nu_1}\ln\frac{1}{M_1}$. Since $\beta(+\infty) = \beta_0$, we can choose M_1 sufficiently large such that $\beta(-\xi) \geq \beta_0 - \epsilon$ for $\xi < \zeta_2 < 0$. If $\xi > \zeta_2$, then $\psi_-(\xi) = 0$ and the fourth inequality of Definition 3.1 holds naturally. If $\xi < \zeta_2$, then $\psi_-(\xi) = \delta(1 - M_1 e^{\nu_1\xi})$. Since $c, \tau > 0$, we have $\psi_-(\xi) \leq \psi_-(\xi - c\tau) \leq \delta$ for all $\xi \in \mathbb{R}$. By monotonicity, we have

$$\frac{\psi_-(\xi)}{1+\alpha\psi_-(\xi)} \leq \frac{\psi_-(\xi - c\tau)}{1+\alpha\psi_-(\xi - c\tau)}.$$

Meanwhile, by the definition of $\psi_-(\xi)$, we have

$$\int_{\mathbb{R}} J(y)\psi_-(\xi - y)\mathrm{d}y \geq \max\{\delta - \delta M_1 e^{\nu_1\xi}\int_{\mathbb{R}} J(y)e^{-\nu_1 y}\mathrm{d}y,\ 0\}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}} J(y)\psi_-(\xi - y)\mathrm{d}y - \psi_-(\xi) - c\psi_-'(\xi) + \frac{\beta(-\xi)\phi_-(\xi)\psi_-(\xi - c\tau)}{1+\alpha\psi_-(\xi - c\tau)} - (\mu+\gamma)\psi_-(\xi)\\
\geq&\delta M_1 e^{\nu_1\xi}\left[-\int_{\mathbb{R}} J(y)\left[e^{-\nu_1 y} - 1\right]\mathrm{d}y + c\nu_1\right] + \frac{(\beta_0 - \epsilon)\phi^*\psi_-(\xi)}{1+\alpha\psi_-(\xi)} - (\mu+\gamma)\psi_-(\xi)\\
\geq&\frac{(\beta_0 - \epsilon)\phi^*\psi_-(\xi)}{1+\alpha\delta} - (\mu+\gamma)\psi_-(\xi)\\
=&\psi_-(\xi)\left[\frac{(\beta_0 - \epsilon)\phi^*}{1+\alpha\delta} - (\mu+\gamma)\right] \geq 0.
\end{aligned}$$

Hence the last inequality of Definition 3.1 holds. We complete the proof of the lemma. □

Proof. [**Proof of Theorem 2.2.**] In view of Lemmas 3.2, 3.3 and 4.3, we see that for each $c > 0$, there exists a solution (ϕ, ψ) of (2.2) satisfying $(\phi, \psi)(+\infty) = (1, 0)$. Next we show $(\phi, \psi)(-\infty) = (S^*, I^*)$.

From the construction of the subsolution (ϕ_-, ψ_-) in Lemma 4.3, we have

$$\phi_{-\infty} := \liminf_{\xi\to-\infty}\phi(\xi) \geq \phi^* > 0,\ \ \psi_{-\infty} := \liminf_{\xi\to-\infty}\psi(\xi) \geq \delta > 0.$$

And we denote

$$\phi_{+\infty} := \limsup_{\xi\to-\infty}\phi(\xi),\ \ \psi_{+\infty} := \limsup_{\xi\to-\infty}\psi(\xi).$$

Define the following functions for $\nu \in [0,1]$

$$h_1(\nu) := \nu S^*, \quad H_1(\nu) := \nu S^* + (1-\nu)(1+Y),$$
$$h_2(\nu) := \nu I^* + (1-\nu)\left(-\frac{1}{\alpha}\right), \quad H_2(\nu) := \nu I^* + (1-\nu)(\psi^* + Z),$$

where $Y = \frac{k\beta_0 S^*}{\alpha\sigma(1+\alpha I^*)}$ and $Z = \frac{kY(1+\alpha I^*)}{\alpha S^*}$ for some constant k with $k > 1$.

For $\nu = 0$, there holds that

$$h_1(\nu) < \phi_{-\infty} \le \phi_{+\infty} < H_1(\nu), \ \ h_2(\nu) < \psi_{-\infty} \le \psi_{+\infty} < H_2(\nu). \tag{4.1}$$

Thus we can define the following quantity

$$\nu^* := \limsup\{\nu \in [0,1) : (4.1) \text{ is true}\} \in [0,1].$$

Note that h_i is increasing in $\nu \in [0,1]$ and H_i is decreasing in $\nu \in [0,1]$ for $i = 1,2$. Since $h_1(1) = H_1(1) = S^*$ and $h_2(1) = H_2(1) = I^*$, if we can show $\nu^* = 1$, then the desired conclusion follows. For contradiction, we assume that $\nu^* < 1$. Letting $\nu \to \nu^*$ in (4.1), we have

$$h_1(\nu) \le \phi_{-\infty} \le \phi_{+\infty} \le H_1(\nu), \ \ h_2(\nu) \le \psi_{-\infty} \le \psi_{+\infty} \le H_2(\nu).$$

In view of the definition of ν^* and the continuity of $h_i(\nu), H_i(\nu)$ for $i = 1,2$, (4.1) cannot be satisfied as $\nu = \nu^*$. Hence at least one of the following equalities holds

$$h_1(\nu^*) = \phi_{-\infty}, \ \ H_1(\nu^*) = \phi_{+\infty}, \ \ h_2(\nu^*) = \psi_{-\infty}, \ \ H_2(\nu^*) = \psi_{+\infty}.$$

(i) Suppose that $h_1(\nu^*) = \phi_{-\infty}$. If ϕ is oscillatory at $-\infty$, then there exists a sequence $\{\xi_n\}$ with $\xi_n \to -\infty$ as $n \to +\infty$ such that $\phi'(\xi_n) = 0$ and $\lim_{n\to+\infty} \phi(\xi_n) = h_1(\nu^*)$. By Fatou's lemma, we have

$$\begin{aligned}\liminf_{n\to+\infty}\left\{\int_{\mathbb{R}} J(y)\phi(\xi_n - y)\mathrm{d}y\right\} &\ge \int_{\mathbb{R}} \liminf_{n\to+\infty} J(y)\phi(\xi_n - y)\mathrm{d}y \\ &= \int_{\mathbb{R}} J(y)h_1(\nu^*)\mathrm{d}y = h_1(\nu^*).\end{aligned}$$

Since $0 < \nu^* < 1$, we obtain that

$$1 + \alpha\left[\nu^* I^* + (1-\nu^*)(\psi^* + Z)\right] \ge \nu^*(1+\alpha I^*).$$

Then from the first equation of (2.2), we have

$$\begin{aligned}0 =& \liminf_{n\to+\infty}\left\{d\left[\int_{\mathbb{R}} J(y)\phi(\xi_n - y)\mathrm{d}y - \phi(\xi_n)\right] + \sigma - \sigma\phi(\xi_n)\right. \\ &\left. - \frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)}\right\} \\ \ge& d\left[\liminf_{n\to+\infty}\int_{\mathbb{R}} J(y)\phi(\xi_n - y)\mathrm{d}y - h_1(\nu^*)\right] + \sigma - \sigma\nu^* S^* \\ &- \frac{\beta_0\nu^* S^*[\nu^* I^* + (1-\nu^*)(\psi^* + Z)]}{1+\alpha[\nu^* I^* + (1-\nu^*)(\psi^* + Z)]} \\ \ge& (1-\nu^*)\sigma + \nu^*(\sigma - \sigma S^*) - \frac{\nu^*\beta_0\nu^* S^* I^*}{\nu^*(1+\alpha I^*)} - \frac{\nu^*\beta_0(1-\nu^*)S^*(\psi^* + Z)}{\nu^*(1+\alpha I^*)} \\ =& (1-\nu^*)\left[\sigma - (\mu+\gamma)(\psi^* + Z)\right].\end{aligned}$$

Since $\psi^* + Z$ is decreasing with respect to α and tends to zero as $\alpha \to +\infty$, it follows that there exists α^* large enough such that $\sigma - (\mu+\gamma)(\psi^* + Z) > 0$ for $\alpha > \alpha^*$. Hence, we derive a contradiction.

On the other hand, we suppose that ϕ is eventually monotone. Then there exists a sequence $\{\xi_n\}$ with $\xi_n \to -\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} \phi(\xi_n) = h_1(\nu^*)$. Similar to the above, there holds that

$$\liminf_{n\to+\infty} \left\{ \sigma - \sigma\phi(\xi_n) - \frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)} \right\} > 0.$$

By integrating the first equation of (2.2) from 0 to ξ_n, we have

$$\begin{aligned} c\left[\phi(\xi_n) - \phi(0)\right] + d\int_{\mathbb{R}} J(y)y\int_0^1 \left[\phi(\xi_n - sy) - \phi(-sy)\right] \mathrm{d}s\mathrm{d}y \\ = \int_0^{\xi_n} \left[\sigma - \sigma\phi(\xi) - \frac{\beta(-\xi)\phi(\xi)\psi(\xi - c\tau)}{1+\alpha\psi(\xi - c\tau)}\right] \mathrm{d}\xi. \end{aligned} \tag{4.2}$$

When n tends to $+\infty$, the left-hand part of (4.2) is uniformly bounded with respect to n, while the right-hand part is unbounded. We get a contradiction and thus $\phi_{-\infty} = h_1(\nu^*)$ cannot happen.

The calculation of the rest cases is similar to that in [11, Theorem 1.3] by using the following inequalities:

(ii) for $\phi_{+\infty} = H_1(\nu^*)$,

$$\begin{aligned} &\limsup_{n\to+\infty} \left\{ \sigma - \sigma\phi(\xi_n) - \frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)} \right\} \\ \le& \sigma - \sigma\left[\nu^* S^* + (1-\nu^*)(1+Y)\right] - \frac{\beta_0\nu^* S^*\left[\nu^* I^* + (1-\nu^*)\left(-\frac{1}{\alpha}\right)\right]}{\nu^*(1+\alpha I^*)} \\ =&(1-\nu^*)\left[-\sigma Y + \frac{\beta_0 S^*}{\alpha(1+\alpha I^*)}\right] < 0; \end{aligned}$$

(iii) for $\psi_{-\infty} = h_2(\nu^*)$: We may assume that $\phi_{-\infty} > h_1(\nu^*)$. Otherwise, it can be reduced to case (i). Then

$$\begin{aligned} &\liminf_{n\to+\infty} \left\{ \frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)} - (\mu+\gamma)\psi(\xi_n) \right\} \\ >& h_2(\nu^*)\left\{ \frac{\beta_0 h_1(\nu^*)}{1+\alpha h_2(\nu^*)} - (\mu+\gamma) \right\} \\ =& h_2(\nu^*)\left\{ \frac{\beta_0 S^*}{1+\alpha I^*} - (\mu+\gamma) \right\} = 0; \end{aligned}$$

(iv) for $\psi_{+\infty} = H_2(\nu^*)$: Note that $1+\alpha I^* - S^* - \alpha S^*\psi^* = 0$ and $\frac{\beta_0 S^*}{1+\alpha I^*} = \mu+\gamma$, then

$$\begin{aligned} &\limsup_{n\to+\infty} \left\{ \frac{\beta(-\xi_n)\phi(\xi_n)\psi(\xi_n - c\tau)}{1+\alpha\psi(\xi_n - c\tau)} - (\mu+\gamma)\psi(\xi_n) \right\} \\ \le& \beta_0 H_2(\nu^*)\left\{ \frac{\nu^* S^* + (1-\nu^*)(1+Y)}{1+\alpha[\nu^* I^* + (1-\nu^*)(\psi^* + Z)]} - \frac{S^*}{1+\alpha I^*} \right\} \\ =& \frac{\beta_0 H_2(\nu^*)(1-\nu^*)}{(1+\alpha I^*)\{1+\alpha[\nu^* I^* + (1-\nu^*)(\psi^* + Z)]\}}\left\{(1+\alpha I^*)Y - \alpha S^* Z\right\} < 0. \end{aligned}$$

The proof is complete. □

5. Nonexistence of forced waves

In this section, we prove the non-existence of forced waves for $\mathcal{R}_0 < 1$.
Proof. [**Proof of Theorem 2.3.**] For contradiction, we assume that (2.2) has a nontrivial nonnegative solution (ϕ, ψ). Since $\phi < 1$, $\psi \geq 0$ and $\beta(-\xi) \leq \beta_0$ in $\mathbb{R}$, it follows that

$$\begin{aligned} c\psi'(\xi) &= \int_{\mathbb{R}} J(y)\psi(\xi - y)\mathrm{d}y - \psi(\xi) + \frac{\beta(-\xi)\phi(\xi)\psi(\xi - c\tau)}{1 + \alpha\psi(\xi - c\tau)} - (\mu + \gamma)\psi(\xi) \\ &\leq \int_{\mathbb{R}} J(y)\psi(\xi - y)\mathrm{d}y - \psi(\xi) + \beta_0\psi(\xi - c\tau) - (\mu + \gamma)\psi(\xi). \end{aligned}$$

Then $I(x, t) = \psi(-x + ct)$ satisfies

$$\begin{cases} \dfrac{\partial I(x,t)}{\partial t} \leq (J * I)(x, t) - I(x, t) + \beta_0 I(x, t - \tau) - (\mu + \gamma)I(x, t), \\ I(x, s) = \psi(-x + cs), \quad x \in \mathbb{R}, \ s \in [-\tau, 0]. \end{cases}$$

Let $u_0 = \sup\limits_{\xi\in\mathbb{R}} \psi(\xi)$. Then it is easy to see that $u_0 > 0$. We consider the following ODE initial value problem

$$\begin{cases} u'(t) = \beta_0 u(t - \tau) - (\mu + \gamma)u(t), \quad t > 0, \\ u(s) \equiv u_0, \quad s \in [-\tau, 0]. \end{cases}$$

By the comparison principle, it follows that

$$I(x, t) \leq u_0 e^{\lambda_0 t}, \ \ t > 0, \tag{5.1}$$

where λ_0 satisfies

$$\lambda_0 = \beta_0 e^{-\lambda_0 \tau} - (\mu + \gamma).$$

Since $\beta_0 < \mu + \gamma$, it is easy to obtain that $\mathrm{Re}\lambda_0 < 0$. Now given $\xi \in \mathbb{R}$, it follows from (5.1) that

$$\psi(\xi) = I(-\xi + ct, t) \leq u_0 e^{\lambda_0 t}, \ \ t > 0.$$

Letting $t \to +\infty$, we have $\psi(\xi) \leq 0$, and hence $\psi \equiv 0$ for $\xi \in \mathbb{R}$, which is a contradiction. We complete the proof. □

6. Discussion

In this section, we summarize the theoretical results of our findings and provide some biological explanations. Due to the shifting effects of the nonlocal dispersal SIR epidemic model, we obtain two types of forced waves connecting $(1, 0)$ to $(1, 0)$ and (S^*, I^*) to $(1, 0)$, respectively. The wave speed c coincides with the shifting speed and is positive. In view of our assumptions, this indicates that climate change will suppress the spread of diseases over time. Meanwhile, at any fixed location x, as time passes and t tends to infinity, the disease will eventually die out (since $\beta(-\infty) = 0$). This kind of forced waves can be viewed as extinction waves for the point-wise "die-out dynamics" [18]. When $c > c^*$, there is a pulse wave connecting $(1, 0)$ to $(1, 0)$ such that the persistence of disease cannot exist. A possible explanation for this phenomenon is that the climate changes so rapidly that pathogens or viruses are unable to survive. When $c > 0$ and $\alpha > \alpha^*$, there is a forced epidemic wave connecting (S^*, I^*) to $(1, 0)$. We conjecture that the large saturation parameter can reduce the impact of environmental shift to a certain extent. Multi-type forced waves reflect the abundant dynamics of nonmonotone systems with shifting effects.

Finally, we consider the effect of time delay τ on wave speed c^*. By Lemma 4.1, we have

$$\Delta(\lambda^*, c^*) = \int_{\mathbb{R}} J(y)e^{-\lambda^* y}\mathrm{d}y - c^*\lambda^* + \beta_0 e^{-\lambda^* c^* \tau} - (\mu + \gamma + 1) = 0.$$

A direct calculation yields that

$$\frac{\mathrm{d}c^*}{\mathrm{d}\tau} = -\frac{\beta_0 c^*}{e^{\lambda^* c^* \tau} + \beta_0 \tau} < 0.$$

Biologically, it suggests that the longer the incubation period of the disease, the slower it spreads. But the slower shifting speed may make the disease take longer to die out in a fixed location.

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